EXISTENCE OF SMALL LOOPS IN A BIFURCATION DIAGRAM NEAR THE DEGENERATE EIGENVALUES

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Abstract. In this paper we study for the incompressible Euler equations the global structure of the bifurcation diagram for the rotating doubly connected patches near the degenerate case. We show that the branches with the same symmetry merge forming a small loop provided that they are close enough. This confirms the numerical observations done in the recent work [10].

Contents

1. Introduction

During the last few decades an intensive research activity has been dedicated to the study in fluid dynamics of relative equilibria, sometimes called steady states or V-states. These vortical structures have the common feature to keep their shape without deformation during the motion and they seem to play a central role in the emergence of coherent structures in turbulent flows at large scales, see for instance [12, 21, 23, 24, 25] and the references therein. Notice that from experimental standpoint, their existence has been revealed in different geophysical phenomena such as the aerodynamic trailing-vortex problem, the two-dimensional shear layers, Saturn's hexagon, the Kármán vortex street, and so on.. Several numerical and analytical investigations have been carried out in various configurations depending on the topological structure of the vortices: single simply or multiply connected vortices, dipolar or multipolar, see for instance [4, 5, 9, 10, 11, 13, 15, 16, 17, 18, 20].
In this paper we shall be concerned with some refined global structure of the doubly connected rotating patches for the two-dimensional incompressible Euler equations. These equations

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describe the motion of an ideal fluid and take the form,

$$
(1.1) \qquad \begin{cases} \partial_t \omega + v \cdot \nabla \omega = 0, \ (t,x) \in \mathbb{R}_+ \times \mathbb{R}^2, \\ v = -\nabla^\perp (-\Delta)^{-1} \omega, \\ \omega_{|t=0} = \omega_0 \end{cases}
$$

where $v = (v_1, v_2)$ refers to the velocity fields and ω being its vorticity which is defined by the scalar $\omega = \partial_1 v_2 - \partial_2 v_1$. Note that one can recover the velocity from the vorticity distribution according to the Biot-Savart law,

$$
v(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{(x-y)^\perp}{|x-y|^2} \omega(y) dy.
$$

The global existence and uniqueness of solutions with initial vorticity lying in the space $L^1 \cap L^\infty$ is a very classical fact established many years ago by Yudovich [29]. This result has the advantage to allow discontinuous vortices taking the form of vortex patches, that is $\omega_0(x) = \chi_D$ the characteristic function of a bounded domain D. The time evolution of this specific structure is preserved and the vorticity $\omega(t)$ is uniformly distributed in bounded domain D_t, which is nothing but the image by the flow mapping of the initial domain. The regularity of this domain is not an easy task and was solved by Chemin in [7] who proved that a $C^{1+\epsilon}$-boundary keeps this regularity globally in time without any loss. In general the dynamics of the boundary is hard to track and is subject to the nonlinear effects created by the induced velocity. Nonetheless, some special family of rotating patches characterized by uniform rotation without changing the shape are known in the literature and a lot of implicit examples have been discovered in the last few decades. Note that in this setting we have explicitly $D_t = R_{0,\Omega t} D$ where $R_{0,\Omega t}$ is a planar rotation centered at the origin and with angle Ωt; for the sake of simplicity we have assumed that the center of rotation is the origin of the frame and the parameter Ω denotes the angular velocity of the rotating domains. The first example was discovered very earlier by Kirchhoff in [22] who showed that an ellipse of semi-axes a and b rotates about its center uniformly with the angular velocity $\Omega = \frac{ab}{(a^2+b^2)}$. Later, Deem and Zabusky gave in [9] numerical evidence of the existence of the V-states with $m-$fold symmetry for the integers $m \in \{3,4,5\}$. Few years after, Burbea gave in [2] an analytical proof of the existence using complex analysis formulation and bifurcation theory. The regularity of the V-states close to Rankine vortices was discussed quite recently in [4, 16]. We point out that the bifurcation from the ellipses was studied numerically and analytically in [5, 20, 21]. All these results are restricted to simply connected domains and the analytical investigation of doubly connected V-states has been initiated with the works [10, 17]. To fix the terminology, a domain D is said doubly connected if it takes the form $D = D_1 \setminus D_2$ with D_1 and D_2 being two simply connected bounded domains satisfying $\overline{D_2} \subset D_1$. The main result of [10] which is deeply connected to the aim of this paper deals with the bifurcation from the annular patches where $D = \mathbb{A}_b \equiv \{z; b < |z| < 1\}$. For the clarity of the discussion we shall recall the main result of [10].

Theorem 1.1. *Given $b \in (0,1)$ and let $m \geq 3$ be a positive integer such that,*

$$
(1.2) \qquad 1 + b^m - \frac{(1-b^2)}{2} m < 0.
$$

Then there exist two curves of doubly connected rotating patches with m-fold symmetry bifurcating from the annulus $\mathbb{A}_b$ at the angular velocities,

$$\Omega_m^{\pm} = \frac{1-b^2}{4} \pm \frac{1}{2m}\sqrt{\Delta_m}$$

with

$$\Delta_m = \left(\frac{1-b^2}{2}m - 1\right)^2 - b^{2m}.$$

We emphasize that the condition (1.2) is required by the transversality assumption, otherwise the eigenvalues $\Omega_m^{\pm}$ are double and thus the classical theorems in the bifurcation theory such as Crandall-Rabinowitz theorem [8] are out of use. The analysis of the degenerate case corresponding to vanishing discriminant (in which case $\Omega_m^+ = \Omega_m^-$) has been explored very recently in [18]. They proved in particular that for $m \geq 3$ and $b \in (0,1)$ such that $\Delta_m = 0$ there is no bifurcation to m-fold V-states. However for $b \in (0,1)\backslash\mathcal{S}$ two-fold V-states still bifurcate from the annulus $\mathbb{A}_b$ where $\mathcal{S} = \{b_m^*, m \geq 3\}$ and b_m^* being the unique solution in the interval $(0,1)$ of the equation

$$1 + b^m - \frac{(1-b^2)}{2}m = 0. \tag{1.3}$$

The proof of this result is by no means non trivial and based on the local structure of the reduced bifurcation equation obtained through the use of Lyapunov-Schmidt reduction. Note that according to the numerical experiments done in [10] two different scenarios for global bifurcation are conjectured. The first one when the eigenvalues Ω_m^- and Ω_m^+ are far enough in which case each branch ends with a singular V-state and the singularity is a corner of angle $\frac{\pi}{2}$. For more details about the structure of the limiting V-states we refer the reader to the Section 9.3 in [10]. Nevertheless, in the second scenario where the eigenvalues are close enough there is no singularity formation on the boundary and it seems quite evident that there is no spectral gap and the V-states can be constructed for any $\Omega \in [\Omega_m^-, \Omega_m^+]$, see Fig. 1 taken from [10]. Moreover, drawing the second Fourier coefficient of each conformal mapping that parametrize each boundary as done in Fig. 2 we get a small loop passing through the trivial solution at Ω_m^- and Ω_m^+. This suggests that the bifurcation curve starting from Ω_m^+ will return back to the trivial solution (annulus) at Ω_m^-.
Our main purpose in this paper is to go further in this study by checking analytically the second scenario and provide for $m \geq 3$ the global structure of the bifurcation curves near the degenerate case. Our result reads as follows.

Theorem 1.2. *Let $m \geq 3$ and b_m^* be the unique solution in $(0,1)$ of the equation (1.3). Then there exists $b_m \in (0, b_m^*)$ such that for any b in (b_m, b_m^*) the two curves of m-fold V-states given by Theorem 1.1 merge and form a loop.*

Now, we are going to outline the main steps of the proof. Roughly speaking we start with writing down the equations governing the boundary $\partial D_1 \cup \partial D_2$ of the rotating patches and attempt to follow the approach developed in [10]. For $j \in \{1,2\}$, let $\Phi_j : \mathbb{D}^c \to D_j^c$ be the conformal mapping which enjoys the following structure,

$$\forall\, |z| \geq 1,\ \Phi_j(z) = b_j z + \sum_{n\in\mathbb{N}} \frac{a_{j,n}}{z^n},\ a_{j,n} \in \mathbb{R},\ b_1 = 1 \quad \text{and} \quad b_2 = b. \tag{1.4}$$

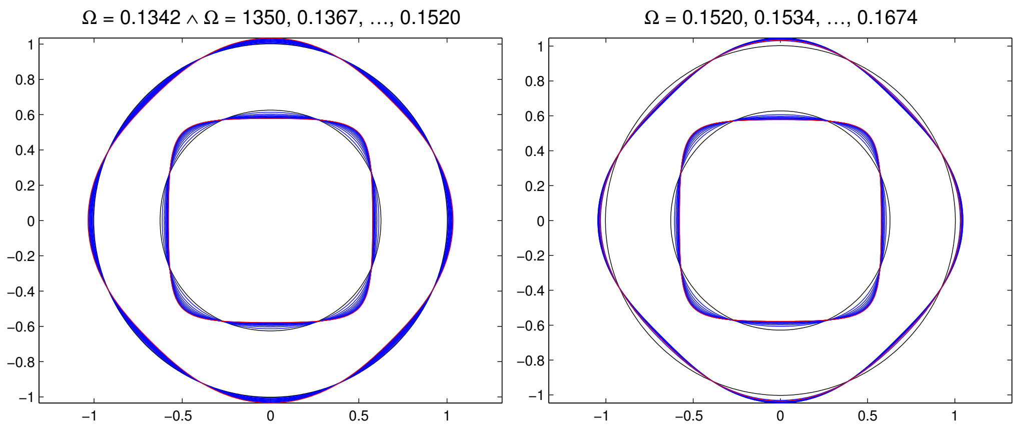

FIGURE 1. Family of 4-fold V-states, for $b = 0.63$ and different Ω. We observe that there is no singularity formation in the boundary.

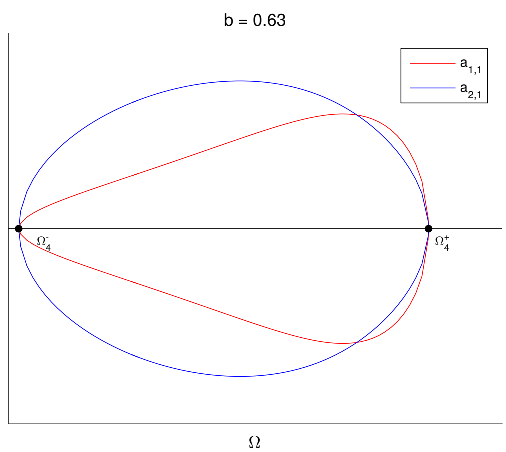

FIGURE 2. Bifurcation curves of the Fourier coefficients $a_{1,1}$ and $a_{2,1}$ in (1.4) with respect to Ω.

We have denoted by $\mathbb{D}^c$ the complement of the open unit disc $\mathbb{D}$ and we have also assumed that the Fourier coefficients of the conformal mappings are real which means that we look for V-states which have at least one axis of symmetry that can be chosen to be the real axis. It is also important to mention that the domain D is implicitly assumed to be smooth enough, more than C^1 as we shall see in the proof, and therefore each conformal mapping can be extended up to the boundary. According to the subsection 2.2 the conformal mappings satisfy the coupled equations: for $j \in \{1, 2\}$

$$G_j(\lambda, f_1, f_2)(w) \triangleq \operatorname{Im}\left\{\Big((1-\lambda)\overline{\Phi_j(w)} + I(\Phi_j(w))\Big)\, w\Phi_j'(w)\right\} = 0\,, \forall w \in \mathbb{T}$$

where

$$\lambda = 1 - 2\Omega, \quad \Phi_j(w) = b_j w + f_j(w)$$

and

$$I(z) = \frac{1}{2i\pi}\int_{\mathbb{T}} \frac{\overline{z} - \overline{\Phi_1(\xi)}}{z - \Phi_1(\xi)}\Phi_1'(\xi)d\xi - \frac{1}{2i\pi}\int_{\mathbb{T}} \frac{\overline{z} - \overline{\Phi_2(\xi)}}{z - \Phi_2(\xi)}\Phi_2'(\xi)d\xi.$$

Here $d\xi$ denotes the complex integration over the unit circle $\mathbb{T}$. The linearized operator around the annulus defined through

$$\mathcal{L}_{\lambda,b}(h) \triangleq \partial_f G(\lambda,0)h = \frac{d}{dt}[G(\lambda,th)]_{|t=0}$$

plays a significant role in the proof and according to [18] it acts as a matrix Fourier multiplier. Actually, for $h=(h_1,h_2)$ chosen in suitable Banach space with

$$h_j(w) = \sum_{n\geq 1} \frac{a_{j,n}}{w^{nm-1}},\ a_{j,n}\in\mathbb{R}$$

we have the expression

$$\mathcal{L}_{\lambda,b}(h) = \sum_{n\geq 1} M_{nm}(\lambda)\begin{pmatrix} a_{1,n} \\ a_{2,n}\end{pmatrix} e_{nm} \quad \text{with} \quad e_n(w)=\operatorname{Im}(\overline{w}^n)$$

where for $n\geq 1$ the matrix M_n is given by

$$M_n(\lambda) = \begin{pmatrix} n\lambda - 1 - nb^2 & b^{n+1} \\ -b^n & b(n\lambda - n + 1)\end{pmatrix}.$$

It is known from [10] that for given $m\geq 3$ the values of b such that $M_m(\lambda)$ is singular, for suitable values of $\lambda=\lambda_m^{\pm}$, belong to the interval $(0,b_m^*)$ where b_m^* has been introduced in (1.3). It is also shown in that paper that the assumptions of Crandall-Rabinowitz theorem are satisfied, especially the transversality assumption which reduces the bifurcation study to some properties of the linearized operator. This latter property is no longer true for $b=b_m^*$ and we have double eigenvalues $\lambda_m^{\pm}=\frac{1+b_m^*}{2}$. This is a degenerate case and we know from [18] that there is no bifurcation. It seems that the approach implemented in this situation can be carried out for $b\in(0,b_m^*)$ but close enough to b_m^*. In fact, using Lyapunov-Schmidt reduction (through appropriate projections) we transform the infinite-dimensional problem into a two dimensional one. Therefore the V-states equation reduces to the resolution of an equation of the type

$$F_2(\lambda,t)=0 \quad \text{with} \quad F_2:\mathbb{R}^2\to\mathbb{R},$$

with F_2 being a smooth function and note that when $b=b_m^*$ the point $(\lambda_m^{\pm},0)$ is a critical point for F_2 and for that reason one should expand F_2 to the second order around this point in order to understand the resolvability of the reduced equation. At the order two F_2 is strictly convex and therefore locally the critical point is the only solution for F_2. Reproducing this approach in the current setting and after long and involved computations we find that for (λ,t) close enough, for example, to the solution $(\lambda_m^+,0)$

$$F_2(\lambda,t) = a_m(b)(\lambda-\lambda_m^+) + c_m(b)(\lambda-\lambda_m^+)^2 + d_m(b)t^2 + \big((\lambda-\lambda_m^+)^2+t^2\big)\varepsilon(\lambda,t)$$

with

$$\lim_{(\lambda,t)\to(\lambda_m^+,0)} \varepsilon(\lambda,t)=0.$$

Notice that

$$a_m(b_m^*)=0, \quad c_m(b_m^*)>0, \quad d_m(b_m^*)>0$$

and moreover for b belonging to a small interval (b_m,b_m^*) we get $a_m(b)>0$. As we can easily check from the preceding facts, the zeros of the associated quadratic form of F_2 is a small ellipse. Therefore by perturbation arguments, we may show that the solutions of F_2 are

actually a perturbation in a strong topology of the ellipse. Consequently the solutions of F_2 around the point $(\lambda_m^+, 0)$ can be parametrized by a smooth Jordan curve. We remark that in addition to the known trivial solution $(\lambda_m^+, 0)$ one can find a second one, of course different from the preceding one, in the form $(\lambda_m, 0)$, corresponding geometrically to the same annulus $\mathbb{A}_b$. Around this point $(\lambda_m, 0)$ the obtained curve describes a bifurcating curve of V-states with exactly $m-$fold symmetry and therefore from the local description of the bifurcation diagram stated in Theorem 1.1 we deduce that $\lambda_m = \lambda_m^-$. This means the formation of loops as stated in our main theorem.
The paper is organized as follows. In Section 2 we shall introduce some tools, formulate the V-states equations and write down the reduced bifurcation equation using Lyapunov-Schmidt reduction. Section 3 is devoted to Taylor expansion at order two of the reduced bifurcation equation and the complete proof of Theorem 1.2 will be given in the last section.

2. Reminder and preliminaries

We shall recall in this section some tools that we shall frequently use throughout the paper and write down the reduced bifurcation equation which is the first step towards the proof of Theorem 1.2. But before that we will fix some notations. The unit disc and its boundary will be denoted respectively by $\mathbb{D}$ and $\mathbb{T}$. and D_r is the planar disc of radius r and centered at the origin. Given a continuous function $f : \mathbb{T} \to \mathbb{C}$ we define its mean value by,

$$\fint_{\mathbb{T}} f(\tau)d\tau \triangleq \frac{1}{2\mathrm{i}\pi}\int_{\mathbb{T}} f(\tau)d\tau,$$

where $d\tau$ stands for the complex integration.

2.1. Hölder spaces.

Now we shall introduce Hölder spaces on the unit circle $\mathbb{T}$. Let $0 < \gamma < 1$ we denote by $C^\gamma(\mathbb{T})$ the space of continuous functions f such that

$$\|f\|_{C^\gamma(\mathbb{T})} \triangleq \|f\|_{L^\infty(\mathbb{T})} + \sup_{\tau\neq w\in\mathbb{T}} \frac{|f(\tau)-f(w)|}{|\tau-w|^\alpha} < \infty.$$

For any integer n, the space $C^{n+\gamma}(\mathbb{T})$ stands for the set of functions f of class C^n whose $n-$th order derivatives are Hölder continuous with exponent γ. It is equipped with the usual norm,

$$\|f\|_{C^{n+\gamma}(\mathbb{T})} \triangleq \|f\|_{L^\infty(\mathbb{T})} + \Big\|\frac{d^n f}{dw^n}\Big\|_{C^\gamma(\mathbb{T})}.$$

Recall that for $n \in \mathbb{N}$, the space $C^n(\mathbb{T})$ is the set of functions f of class C^n such that,

$$\|f\|_{C^n(\mathbb{T})} \triangleq \sum_{k=0}^{n} \|f^{(k)}\|_{L^\infty(\mathbb{T})} < \infty.$$

2.2. Boundary equations.

Let $D_2 \Subset D_1$ be two simply connected domains and $D = D_1 \setminus D_2$ be a doubly connected domains. The boundary of D_j will be denoted by Γ_j. Then according to [10, 18] we find that that the exterior conformal mappings Φ_1 and Φ_2 associated to D_1 and D_2 satisfy the coupled nonlinear equations. For $j \in \{1, 2\}$,

$$\widehat{G}_j(\lambda, \Phi_1, \Phi_2)(w) = 0, \quad \forall\, w \in \mathbb{T},$$

with

$$\widehat{G}_j(\lambda,\Phi_1,\Phi_2)(w)\triangleq \operatorname{Im}\Big\{\Big((1-\lambda)\overline{\Phi_j(w)}+I(\Phi_j(w))\Big)\,w\Phi_j'(w)\Big\}. \tag{2.1}$$

Note that we have introduced $\lambda \triangleq 1-2\Omega$ because it is more convenient for the computations and

$$I(z)=\fint_{\mathbb{T}}\frac{\overline{z}-\overline{\Phi_1(\xi)}}{z-\Phi_1(\xi)}\Phi_1'(\xi)d\xi-\fint_{\mathbb{T}}\frac{\overline{z}-\overline{\Phi_2(\xi)}}{z-\Phi_2(\xi)}\Phi_2'(\xi)d\xi.$$

The integrals are defined in the complex sense and we shall focus on V-states which are small perturbation of the annulus $\mathbb{A}_b=\big\{z, b\le |z|\le 1\big\}$ with $b\in(0,1)$. The conformal mappings Φ_j with $j\in\{1,2\}$ admit the expansions,

$$\forall |z|\ge 1,\quad \Phi_j(z)=b_jz+f_j(z)=z+\sum_{n=1}^{+\infty}\frac{a_{j,n}}{z^n}$$

with

$$b_1=1, b_2=b.$$

Define

$$G_j(\lambda,f_1,f_2)\triangleq \widehat{G}_j(\lambda,\Phi_1,\Phi_2) \tag{2.2}$$

then the equations of the V-states become,

$$\forall w\in\mathbb{T},\; G(\lambda,f_1,f_2)(w)=0$$

with

$$G=(G_1,G_2).$$

Note that the annulus is a solution for any angular velocity, that is,

$$G(\lambda,0,0)=0$$

and the set $\{(\lambda,0,0)|\lambda\in\mathbb{R}\}$ will be called the set of trivial solutions.

2.3. **Reduced bifurcation equation.** For any integer $m\ge 3$, the existence of V-states was proved in [10] provided $b\in(0,b_m^*)$ that guarantee the transversality assumption. The idea is to check that the functional G has non trivial zeros using bifurcation arguments. However, and as we have mentioned before in the Introduction, the knowledge of the linearized operator around the trivial solution is not enough to understand the structure of the bifurcating curves near the degenerate case corresponding to double eigenvalues. To circumvent this difficulty we make an expansion at order two of the reduced bifurcation equation in the spirit of [18], and this will be the subject of the current task. Let us first introduce Banach spaces that we shall use and recall the algebraic structure of the linearize operator. For $\alpha\in(0,1)$, we set

$$X_m=\left\{f=(f_1,f_2)\in (C^{1+\alpha}(\mathbb{T}))^2, f(w)=\sum_{n=1}^{+\infty}A_n\overline{w}^{nm-1}, A_n\in\mathbb{R}^2\right\}.$$

and

$$Y_m=\left\{G=(G_1,G_2)\in(C^{\alpha}(\mathbb{T}))^2, G=\sum_{n=1}^{+\infty}B_ne_{nm}, B_n\in\mathbb{R}^2\right\},\; e_n(w)=\operatorname{Im}(\overline{w}^n).$$

Note that the domains D_j whose conformal mappings Φ_j associated to the perturbations f_j lying in X_m are actually $m-$fold symmetric. Recall from the subsection 2.2 that the equation of $m-$fold symmetric V-states is given by

$$G(\lambda, f) = 0, \quad f = (f_1, f_2) \in B_r^m \times B_r^m \subset X_m \tag{2.3}$$

where B_r^m is the ball given by

$$B_r^m = \Big\{ f \in C^{1+\alpha}(\mathbb{T}),\ f(w) = \sum_{n=1}^{\infty} a_n \overline{w}^{nm-1},\ a_n \in \mathbb{R}, \|f\|_{C^{1+\alpha}} \leq r \Big\}.$$

We mention that we are looking for solutions close to the trivial solutions and therefore the radius r will be taken small enough. The linearized operator around zero is defined by

$$\partial_f G(\lambda, 0)h = \frac{d}{dt}[G(\lambda, th)]_{|t=0}.$$

As it is proved in [18], for $h = (h_1, h_2) \in X_m$ taking the expansions

$$h_j(w) = \sum_{n \geq 1} \frac{a_{j,n}}{w^{nm-1}},$$

we get the expression

$$\partial_f G(\lambda, 0)h = \sum_{n \geq 1} M_{nm}(\lambda) \begin{pmatrix} a_{1,n} \\ a_{2,n} \end{pmatrix} e_{nm}, \tag{2.4}$$

where for $n \geq 1$ the matrix M_n is given by

$$M_n(\lambda) = \begin{pmatrix} n\lambda - 1 - nb^2 & b^{n+1} \\ -b^n & b(n\lambda - n + 1) \end{pmatrix}.$$

We say throughout this paper that λ is an eigenvalue if for some n the matrix $M_n(\lambda)$ is not invertible. Since

$$\det(M_n(\lambda)) = (n\lambda - 1 - nb^2)b(n\lambda - n + 1) + b^{2n+1}$$

is a polynomial of second order on the variable λ, the roots are real if and only if its discriminant is positive. From [18] we remind that the roots take the form,

$$\lambda_n^{\pm} = \frac{1+b^2}{2} \pm \frac{1}{n}\sqrt{\Delta_n(b)}$$

with the constraint

$$\Delta_n(b) = \Big(\frac{1-b^2}{2}n - 1\Big)^2 - b^{2n} \geq 0.$$

According to [10] this condition is equivalent for $n \geq 3$ to

$$n\frac{1-b^2}{2} - 1 \geq b^n. \tag{2.5}$$

In addition, it is also proved that for any integer $m \geq 3$ there exists a unique $b_m^* \in (0,1)$ such that $\Delta_m(b_m^*) = 0$ and $\Delta_m(b) > 0$ for all $b \in [0, b_m^*)$. Moreover,

$$\mathrm{Ker}(\partial_f G(\lambda_m^{\pm}, 0)) = \langle v_m \rangle$$

with

$$v_m(w)=\left(\begin{array}{c}\frac{m\lambda_m^\pm-m+1}{b^{m-1}}\\ 1\end{array}\right)\overline{w}^{m-1}\triangleq\left(\begin{array}{c}v_{1,m}\\ v_{2,m}\end{array}\right)\overline{w}^{m-1}.$$

In order to be rigorous we could write $v_m^\pm$ but for the sake of simple notations we note simply v_m. Now we shall introduce a complement $\mathcal{X}_m$ of the subspace $\langle v_m\rangle$ in the space X_m,

$$\mathcal{X}_m=\left\{h\in(C^{1+\alpha}(\mathbb{T}))^2, h(w)=\sum_{n=2}^{+\infty}A_n\overline{w}^{nm-1}+\alpha\left(\begin{array}{c}1\\ 0\end{array}\right)\overline{w}^{m-1}, A_n\in\mathbb{R}^2,\alpha\in\mathbb{R}\right\}.$$

It is easy to prove that the subspace is closed and

$$X_m=\langle v_m\rangle\oplus\mathcal{X}_m.$$

In addition the range $\mathcal{Y}_m$ of $\partial_f G(\lambda_m^\pm,0)$ in Y_m is given by

$$\mathcal{Y}_m=\left\{K\in(C^{\alpha}(\mathbb{T}))^2, K=\sum_{n=2}^{+\infty}B_ne_{nm}+\beta\left(\begin{array}{c}b^m\\ m\lambda_m^\pm-m+1\end{array}\right)e_m, B_n\in\mathbb{R}^2,\beta\in\mathbb{R}\right\}.$$

The subspace $\mathcal{Y}_m$ is of co-dimension one and its complement is a line generated by

$$\begin{aligned}\mathbb{W}_m&=\frac{1}{\sqrt{(m\lambda_m^\pm-m+1)^2+b^{2m}}}\left(\begin{array}{c}m\lambda_m^\pm-m+1\\ -b^m\end{array}\right)e_m\\ &\triangleq\widehat{\mathbb{W}}_me_m.\end{aligned}$$

Thus we have

$$Y_m=\langle\mathbb{W}_m\rangle\oplus\mathcal{Y}_m.$$

Lyapunov-Schmidt reduction relies on two projections

$$P:X_m\to\langle v_m\rangle,\quad Q:Y_m\to\langle\mathbb{W}_m\rangle.$$

For a future use we need the explicit expression of the projection Q. The Euclidian scalar product of $\mathbb{R}^2$ is denoted by $\langle\,,\,\rangle$ and for $h\in Y_m$ we have

$$h=\sum_{n=1}^{+\infty}B_ne_{nm},\quad Qh(w)=\langle B_1,\widehat{\mathbb{W}}_m\rangle\mathbb{W}_m.$$

Moreover, by the definition of Q one has,

$$Q\partial_fG(\lambda_m^\pm,0)=0. \tag{2.6}$$

Unlike the degenerate case, the transversality assumption holds true

$$\partial_\lambda\partial_fG(\lambda_m^\pm,0)v_m\notin\mathrm{Im}(\partial_fG(\lambda_m^\pm,0))$$

and therefore

$$Q\partial_\lambda\partial_fG(\lambda_m^\pm,0)v_m\neq0. \tag{2.7}$$

For $f\in X_m$ we use the decomposition

$$f=g+k\quad\text{with}\quad g=Pf\quad\text{and}\quad k=(\mathrm{Id}-P)f.$$

Then the V-state equation is equivalent to the system

$$F_1(\lambda,g,k)\triangleq(\mathrm{Id}-Q)G(\lambda,g+k)=0\quad\text{and}\quad QG(\lambda,g+k)=0.$$

Note that $F_1 : \mathbb{R} \times \langle v_m \rangle \times \mathcal{X}_m \to \mathcal{Y}_m$ is well-defined and smooth. Thus using (2.6) we can check the identity,

$$D_k F_1(\lambda_m^\pm, 0, 0) = (\mathrm{Id} - Q)\partial_f G(\lambda_m^\pm, 0) = \partial_f G(\lambda_m^\pm, 0).$$

Consequently

$$D_k F_1(\lambda_m^\pm, 0, 0) : \mathcal{X}_m \to \mathcal{Y}_m$$

is invertible. The inverse is explicit and is given by the formula

$$\partial_f G(\lambda_m^\pm, 0)h = K \Longleftrightarrow \forall n \geq 2, A_n = M_{nm}^{-1} B_n \quad \text{and} \quad \alpha = -\frac{\beta}{b^m}(m\lambda_m^\pm - m + 1)\cdot \tag{2.8}$$

Thus using the implicit function theorem, the solutions of the equation $F_1(\lambda, g, k) = 0$ are locally described around the point $(\lambda_m^\pm, 0)$ by the parametrization $k = \varphi(\lambda, g)$ with

$$\varphi : \mathbb{R} \times \langle v_m \rangle \to \mathcal{X}_m.$$

being a smooth function. Remark that in principle φ is locally defined but it can be extended globally to a smooth function still denoted by φ. Moreover, the resolution of the V-state equation near to $(\lambda_m^\pm, 0)$ is equivalent to

$$QG\big(\lambda, tv_m + \varphi(\lambda, tv_m)\big) = 0. \tag{2.9}$$

As $G(\lambda, 0) = 0$, $\forall \lambda$ it follows

$$\varphi(\lambda, 0) = 0, \; \forall \lambda \in \mathcal{V}(\lambda_m^\pm), \tag{2.10}$$

where $\mathcal{V}(\lambda_m^\pm)$ is a small neighborhood of $\lambda_m^\pm$. Using Taylor expansion at order 1 on the variable t the V-states equation (2.9) is equivalent to the *reduced bifurcation equation*,

$$F_2(\lambda, t) \triangleq \int_0^1 Q\partial_f G(\lambda, stv_m + \varphi(\lambda, stv_m))(v_m + \partial_g \varphi(\lambda, stv_m)v_m)ds = 0. \tag{2.11}$$

In addition, using (2.6) we remark that

$$F_2(\lambda_m^\pm, 0) = 0.$$

3. Taylor expansion

The goal of this section is to compute Taylor expansion of F_2 at the second order. This quadratic form will answer about the local structure of the solutions of the equation (2.11).

3.1. **General formulae.** The aim of this paragraph is to provide some general results concerning the first and second derivatives of φ and F_2. First notice that the transversality assumption required for Crandall-Rabinowitz theorem is given by

$$\partial_\lambda F_2(\lambda_m^\pm, 0) = Q\partial_\lambda \partial_f G(\lambda_m^\pm, 0)v_m \neq 0.$$

Thus applying the implicit function theorem to F_2, we get in a small neighborhood of $(\lambda_m^\pm, 0)$ a unique curve of solutions $t \in [-\varepsilon_0, \varepsilon_0] \mapsto (\lambda(t), t)$. We shall prove that for b close enough to b_m^* this curve a smooth Jordan curve and for this aim we need to know the full structure of the quadratic form associated to F_2. The following identities were proved in [18, p 13-14].

$$\partial_\lambda \varphi(\lambda_m^\pm, 0) = \partial_g \varphi(\lambda_m^\pm, 0)v_m = 0. \tag{3.1}$$

and

$$\partial_{\lambda\lambda} \varphi(\lambda_m^\pm, 0) = 0. \tag{3.2}$$

Now, we give the expressions of the coefficients of the quadratic form associated to F_2 around the point $(\lambda_m^\pm, 0)$. For the proof see [18, Proposition 2].

Proposition 3.1. *The following assertions hold true.*

(1) *First derivatives:*

$$\partial_\lambda F_2(\lambda_m^\pm, 0) = Q\partial_\lambda \partial_f G(\lambda_m^\pm, 0)(v_m)$$

and

$$\begin{aligned}\partial_t F_2(\lambda_m^\pm, 0) &= \frac{1}{2} Q\partial_{ff} G(\lambda_m^\pm, 0)[v_m, v_m] \\ &= \frac{1}{2}\frac{d^2}{dt^2}[QG(\lambda_m^\pm, tv_m)]_{|t=0}.\end{aligned}$$

(2) *Expression of $\partial_{\lambda\lambda} F_2(\lambda_m^\pm, 0)$:*

$$\partial_{\lambda\lambda} F_2(, 0) = -2Q\partial_\lambda \partial_f G(\lambda_m^\pm, 0)[\partial_f G(\lambda_m^\pm, 0)]^{-1}(\mathrm{Id} - Q)\partial_\lambda \partial_f G(\lambda_m^\pm, 0)v_m$$

(3) *Expression of $\partial_{tt} F_2(\lambda_m^\pm, 0)$:*

$$\partial_{tt} F_2(\lambda_m^\pm, 0) = \frac{1}{3}\frac{d^3}{dt^3}[QG(\lambda_m^\pm, tv_m)]_{|t=0} + Q\partial_{ff} G(\lambda_m^\pm, 0)[v_m, \widehat{v}_m]$$

with

$$\begin{aligned}\widehat{v}_m &\triangleq \frac{d^2}{dt^2}\, \varphi(\lambda_m^\pm, tv_m)\Big|_{t=0} \\ &= -\,[\partial_f G(\lambda_m^\pm, 0)]^{-1}\frac{d^2}{dt^2}[(\mathrm{Id} - Q)G(\lambda_m^\pm, tv_m)]_{|t=0}\end{aligned}$$

and

$$Q\partial_{ff} G(\lambda_m^\pm, 0)[v_m, \widehat{v}_m] = \partial_t \partial_s [QG(\lambda_m^\pm, tv_m + s\widehat{v}_m)]_{|t=0,s=0}.$$

(4) *Expression of $\partial_\lambda \partial_t F_2(\lambda_m^\pm, 0)$:*

$$\begin{aligned}\partial_\lambda \partial_t F_2(\lambda_m^\pm, 0) &= \frac{1}{2} Q\partial_\lambda \partial_{ff} G(\lambda_m^\pm, 0)[v_m, v_m] + \frac{1}{2} Q\partial_\lambda \partial_f G(\lambda_m^\pm, 0)(\widehat{v}_m) \\ &\quad + Q\partial_{ff} G(\lambda_m^\pm, 0)[v_m, \partial_\lambda \partial_g \varphi(\lambda_m^\pm, 0)v_m]\end{aligned}$$

with

$$\partial_\lambda \partial_g \varphi(\lambda_m^\pm, 0)v_m = -[\partial_f G(\lambda_m^\pm, 0)]^{-1}(\mathrm{Id} - Q)\partial_\lambda \partial_f G(\lambda_m^\pm, 0)v_m$$

3.2. **Explicit formula for the quadratic form.** In this section we want to explicit the terms in the Taylor expansion of F_2 at the second order. The main result reads as follows.

Proposition 3.2. *Let $m \geq 3$ and $b \in (0, b_m^*)$. Then the following assertions hold true.*

(1) *Expression of $\partial_t F_2(\lambda_m^\pm, 0)$.*

$$\partial_t F_2(\lambda_m^\pm, 0) = 0.$$

(2) *Expression of $\partial_\lambda F_2(\lambda_m^\pm, 0)$.*

$$\partial_\lambda F_2(\lambda_m^\pm, 0) = \frac{m[(m\lambda_m^\pm - m + 1)^2 - b^{2m}]}{b^{m-1}[(m\lambda_m^\pm - m + 1)^2 + b^{2m}]^{\frac{1}{2}}}\mathbb{W}_m.$$

(3) *Expression of* $\partial_{\lambda\lambda}F_2(\lambda_m^\pm,0)$.

$$\partial_{\lambda\lambda}F_2(\lambda_m^\pm,0)=\frac{4m^2b^{1-m}(m\lambda_m^\pm-m+1)^3}{[(m\lambda_m^\pm-m+1)^2+b^{2m}]^{\frac32}}\mathbb{W}_m.$$

(4) *Expression of* $\partial_{tt}F_2(\lambda_m^\pm,0)$.

$$\begin{aligned}\partial_{tt}F_2(\lambda_m^\pm,0)&=-m(m-1)b^{3-3m}\frac{(b^{2m-2}-(m\lambda_m^\pm-m+1)^2)^2}{([m\lambda_m^\pm-m+1]^2+b^{2m})^{\frac12}}\mathbb{W}_m\\&+\widetilde{\beta}_m\mathcal{K}_m\mathbb{W}_m\end{aligned}$$

with

$$\begin{aligned}\mathcal{K}_m&\triangleq\frac{b^{1-m}(m\lambda_m^\pm-1)(m\lambda_m^\pm-m+1)^2+(1-2m)(m\lambda_m^\pm-m+1)b^{m+1}+mb^{3m-1}}{[(m\lambda_m^\pm-m+1)^2+b^{2m}]^{\frac12}}\\&\times\ (2\lambda_m^\pm m-2m+1)\end{aligned}$$

and

$$\widetilde{\beta}_m=-\frac{2bm\big(b^m-b^{2-m}(m\lambda_m^\pm-m+1)\big)^2}{\det(M_{2m}(\lambda_m^\pm))}.$$

(5) *Expression of* $\partial_\lambda\partial_tF_2(\lambda_m^\pm,0)$.

$$\partial_\lambda\partial_tF_2(\lambda_m^\pm,0)=0.$$

Remark 3.3. *In [18], all the preceding quantities were computed in the limit case $b=b_m^*$ and our expressions lead to the same thing when we take $b\to b_m^*$. This can be checked using the identity*

$$m\lambda_m^\pm-m+1=-\sqrt{b^{2m}+\Delta_m}\pm\sqrt{\Delta_m}$$

and when $b=b_m^*$ the discriminant Δ_m vanishes.

In what follows we shall establish the formulae of Proposition 3.2. As we can observe from Proposition 3.1 that most of them are based on the quantities $\frac{d^k}{dt^k}[G(\lambda_m^\pm,tv_m)]_{|t=0}$ for $k\in\{2,3\}$. We introduce some notations which will be very useful to obtain explicit expressions. We begin with:

$$\Phi_j(t,w)=b^{j-1}w+tv_{j,m}\overline{w}^{m-1}$$

wich leads to

$$G_j(\lambda_m^\pm,tv_m)=\operatorname{Im}\Big\{[(1-\lambda_m^\pm)\overline{\Phi_j(t,w)}+I(\Phi_j(t,w))]w\left(b^{j-1}+t(1-m)v_{j,m}\overline{w}^m\right)\Big\}$$

with:

$$I(\Phi_j(t,w))=I_1(\Phi_j(t,w))-I_2(\Phi_j(t,w))$$

where:

$$I_i(\Phi_j(t,w))=\fint_{\mathbb{T}}\frac{\overline{\Phi_j(t,w)}-\overline{\Phi_i(t,\tau)}}{\Phi_j(t,w)-\Phi_i(t,\tau)}\Phi_i'(t,\tau)d\tau.$$

3.2.1. *Computation of* $\partial_t F_2(\lambda_m^\pm, 0)$. We shall sketch the proof because most of the computations were done in [18]. Note that

$$\partial_t F_2(\lambda_m^\pm, 0) = \frac{1}{2} Q \partial_{ff} G(\lambda_m^\pm, 0)[v_m, v_m] = \frac{1}{2} Q \frac{d^2}{dt^2}[G_j(\lambda_m^\pm, tv_m)]_{|t=0}.$$

To lighten the notations we introduce

$$I_i(\Phi_j(t,w)) = \fint_{\mathbb{T}} \frac{\overline{A} + t\overline{B}}{A + tB}(b^{i-1} + tC) d\tau$$

with

$$A = b^{j-1}w - b^{i-1}\tau, \; B = v_{j,m}\overline{w}^{m-1} - v_{i,m}\overline{\tau}^{m-1} \quad \text{and} \quad C = v_{i,m}(1-m)\overline{\tau}^m.$$

We can easily find that

$$\begin{aligned}
\frac{d^2}{dt^2}[G_j(\lambda_m^\pm, tv_m)]_{|t=0} &= \operatorname{Im}\bigg\{ b^{j-1} w \frac{d^2}{dt^2} I(\Phi_j(t,w))_{|t=0} + 2(1-\lambda_m^\pm)(1-m) v_{j,m}^2 \\
&\quad + 2(1-m) v_{j,m} \frac{d}{dt} I(\Phi_j(t,w))_{|t=0} \overline{w}^{m-1} \bigg\} \\
&= \operatorname{Im}\bigg\{ b^{j-1} w \frac{d^2}{dt^2} I(\Phi_j(t,w))_{|t=0} + 2(1-m) v_{j,m} \frac{d}{dt} I(\Phi_j(t,w))_{|t=0} \overline{w}^{m-1} \bigg\}.
\end{aligned}$$

Recall from [18, p. 823] that

$$\frac{d}{dt}[I_i(\Phi_j(t,w))]_{|t=0} = \fint \frac{\overline{A}}{A^2}(AC - b^{i-1}B) d\tau + b^{i-1} \fint \frac{\overline{B}}{A} d\tau.$$

Moreover, for any $i, j \in \{1, 2\}$, there exist real numbers $\mu_{i,j}, \gamma_{i,j}$ such that

$$\fint \frac{\overline{B}}{A} d\tau = \mu_{i,j} w^{m-1}$$

and

$$\fint \frac{\overline{A}}{A^2}(AC - b^{i-1}B) d\tau = \gamma_{i,j} \overline{w}^{m+1}.$$

Hence,

$$\frac{d}{dt}[I_i(\Phi_j(t,w))]_{|t=0} = \gamma_{i,j} \overline{w}^{m+1} + b^{i-1} \mu_{i,j} w^{m-1}$$

with

$$\gamma_{i,j} \triangleq v_{i,m}(1-m) \fint \frac{b^{j-1} - b^{i-1}\overline{\tau}}{b^{j-1} - b^{i-1}\tau} \overline{\tau}^m d\tau - b^{i-1} \fint \frac{b^{j-1} - b^{i-1}\overline{\tau}}{(b^{j-1} - b^{i-1}\tau)^2}(v_{j,m} - v_{i,m}\overline{\tau}^{m-1}) d\tau.$$

We also get from (39) of [18],

$$\gamma_{1,2} = 0.$$

For $\gamma_{2,1}$ by writing

$$\gamma_{2,1} = v_{2,m}(1-m) \fint \frac{1 - b\overline{\tau}}{1 - b\tau} \overline{\tau}^m d\tau - b \fint \frac{1 - b\overline{\tau}}{(1 - b\tau)^2}(v_{1,m} - v_{2,m}\overline{\tau}^{m-1}) d\tau.$$

combined with the following identities: for any $m \in \mathbb{N}^*$

$$\fint_{\mathbb{T}} \frac{\overline{\tau}^m}{(1-b\tau)} d\tau = \fint_{\mathbb{T}} \frac{\tau^{m-1}}{\tau - b} d\tau = b^{m-1} \tag{3.3}$$

and

$$\fint_{\mathbb{T}} \frac{\overline{\tau}^m}{(1-b\tau)^2} d\tau = \fint_{\mathbb{T}} \frac{\tau^m}{(\tau-b)^2} d\tau = mb^{m-1}. \tag{3.4}$$

We obtain

$$\begin{aligned}\gamma_{2,1} &= v_{2,m}(1-m)[b^{m-1} - b^{m+1}] + bv_{2,m}[(m-1)b^{m-2} - mb^m] + b^2 v_{1,m}\\ &= -v_{2,m}b^{m+1} + b^2 v_{1,m}.\end{aligned}$$

For $\gamma_{i,i}$ we recall that

$$\gamma_{i,i} = v_{i,m}(1-m) \fint \frac{1-\overline{\tau}}{1-\tau} \overline{\tau}^m d\tau - v_{i,m} \fint \frac{1-\overline{\tau}}{(1-\tau)^2} (1 - \overline{\tau}^{m-1}) d\tau.$$

Thus using the residue theorem at ∞ we deduce,

$$\gamma_{i,i} = 0.$$

Finally we get,

$$\frac{d}{dt}[I(\Phi_1(t,w))]_{|t=0} = (\mu_{1,1} - b\mu_{2,1})w^{m-1} + [v_{2,m}b^{m+1} - b^2 v_{1,m}]\overline{w}^{m+1} \tag{3.5}$$

and

$$\frac{d}{dt}[I(\Phi_2(t,w))]_{|t=0} = (\mu_{1,2} - b\mu_{2,2})w^{m-1}. \tag{3.6}$$

Now we have to compute $\frac{d^2}{dt^2}[I_i(\Phi_j(t,w))]_{|t=0}$. According to [18, p. 825-826] one has

$$\frac{d^2}{dt^2}[I_i(\Phi_j(t,w))]_{|t=0} = 2 \fint_{\mathbb{T}} \frac{[A\overline{B} - \overline{A}B]}{A^3} [AC - b^{i-1}B] d\tau.$$

Moreover

$$\fint_{\mathbb{T}} \frac{\overline{B}}{A^2} [AC - b^{i-1}B] d\tau = \widehat{\mu}_{i,j} \overline{w}$$

and

$$-\fint_{\mathbb{T}} \frac{\overline{A}B}{A^3} [AC - b^{i-1}B] d\tau = \eta_{i,j} \overline{w}^{2m+1}.$$

Then we have

$$\frac{d^2}{dt^2}[I_i(\Phi_j(t,w))]_{|t=0} = 2\widehat{\mu}_{i,j}\overline{w} + 2\eta_{i,j}\overline{w}^{2m+1}$$

with

$$\widehat{\mu}_{i,j} = v_{i,m}(1-m) \fint_{\mathbb{T}} \frac{(v_{j,m} - v_{i,m}\tau^{m-1})}{(b^{j-1} - b^{i-1}\tau)} \overline{\tau}^m d\tau + \fint_{\mathbb{T}} \frac{(v_{j,m} - v_{i,m}\tau^{m-1})}{(b^{j-1} - b^{i-1}\tau)^2} b^{i-1} [v_{i,m}\overline{\tau}^{m-1} - v_{j,m}] d\tau$$

and

$$\begin{aligned}\eta_{i,j} &= \fint_{\mathbb{T}} \frac{(b^{j-1}-b^{i-1}\overline{\tau})(v_{j,m}-v_{i,m}\overline{\tau}^{m-1})}{(b^{j-1}-b^{i-1}\tau)^2} v_{i,m}(m-1)\overline{\tau}^m d\tau \\ &+ b^{i-1}\fint_{\mathbb{T}} \frac{(b^{j-1}-b^{i-1}\overline{\tau})(v_{j,m}-v_{i,m}\overline{\tau}^{m-1})^2}{(b^{j-1}-b^{i-1}\tau)^3} d\tau.\end{aligned}$$

For the diagonal terms we get

$$\begin{aligned}\widehat{\mu}_{i,i} &= \frac{v_{i,m}^2}{b^{i-1}}(1-m)\fint_{\mathbb{T}} \frac{(1-\tau^{m-1})}{(1-\tau)}\overline{\tau}^m d\tau + \frac{v_{i,m}^2}{b^{i-1}}\fint_{\mathbb{T}} \frac{(1-\tau^{m-1})}{(1-\tau)^2}[\overline{\tau}^{m-1}-1]d\tau \\ &= (m-1)\frac{v_{i,m}^2}{b^{i-1}}\end{aligned}$$

As to the term $\widehat{\mu}_{1,2}$, we may write

$$\begin{aligned}\widehat{\mu}_{1,2} &= v_{1,m}(1-m)\fint_{\mathbb{T}} \frac{(v_{2,m}-v_{1,m}\tau^{m-1})}{(b-\tau)}\overline{\tau}^m d\tau + v_{1,m}\fint_{\mathbb{T}} \frac{(v_{2,m}-v_{1,m}\tau^{m-1})}{(b-\tau)^2}\overline{\tau}^{m-1}d\tau \\ &- v_{2,m}\fint_{\mathbb{T}} \frac{(v_{2,m}-v_{1,m}\tau^{m-1})}{(b-\tau)^2}d\tau.\end{aligned}$$

The first and the second integrals vanish using the residue theorem at ∞. Thus we find

$$\widehat{\mu}_{1,2} = (m-1)v_{2,m}v_{1,m}b^{m-2}.$$

Concerning the term $\widehat{\mu}_{2,1}$ given by

$$\widehat{\mu}_{2,1} = v_{2,m}(1-m)\fint_{\mathbb{T}} \frac{(v_{1,m}-v_{2,m}\tau^{m-1})}{(1-b\tau)}\overline{\tau}^m d\tau + \fint_{\mathbb{T}} \frac{(v_{1,m}-v_{2,m}\tau^{m-1})}{(1-b\tau)^2} b[v_{2,m}\overline{\tau}^{m-1}-v_{1,m}]d\tau$$

it can be computed using (3.3) and (3.4)

$$\begin{aligned}\widehat{\mu}_{2,1} &= v_{2,m}(1-m)[v_{1,m}b^{m-1}-v_{2,m}] + bv_{2,m}v_{1,m}(m-1)b^{m-2} \\ &+ b\fint_{\mathbb{T}} \frac{(-v_{2,m}^2-v_{1,m}^2+v_{2,m}v_{1,m}\tau^{m-1})}{(1-b\tau)^2}d\tau.\end{aligned}$$

The last term vanishes thanks to the residue theorem. Finally we have:

$$\widehat{\mu}_{2,1} = v_{2,m}^2(m-1).$$

Now we shall move to the calculation of $\eta_{i,j}$ for $i,j\in\{1,2\}$. We start with the term,

$$\eta_{1,2} = v_{1,m}(m-1)\fint_{\mathbb{T}} \frac{(b-\overline{\tau})(v_{2,m}-v_{1,m}\overline{\tau}^{m-1})}{(b-\tau)^2}\overline{\tau}^m d\tau + \fint_{\mathbb{T}} \frac{(b-\overline{\tau})(v_{2,m}-v_{1,m}\overline{\tau}^{m-1})^2}{(b-\tau)^3}d\tau.$$

Using the residue theorem at ∞ we get

$$\eta_{1,2} = 0.$$

Now we focus on the term $\eta_{2,1}$ given by

$$\eta_{2,1} = v_{2,m}(m-1)\fint_{\mathbb{T}} \frac{(1-b\overline{\tau})(v_{1,m}-v_{2,m}\overline{\tau}^{m-1})}{(1-b\tau)^2}\overline{\tau}^m d\tau + b\fint_{\mathbb{T}} \frac{(1-b\overline{\tau})(v_{1,m}-v_{2,m}\overline{\tau}^{m-1})^2}{(1-b\tau)^3}d\tau.$$

According to (3.3) and (3.4) we get

$$\begin{aligned}\fint_{\mathbb{T}} \frac{(1-b\overline{\tau})(v_{1,m}-v_{2,m}\overline{\tau}^{m-1})}{(1-b\tau)^2}\overline{\tau}^m d\tau &= v_{1,m}\Big(mb^{m-1}-(m+1)b^{m+1}\Big)\\ &+ v_{2,m}\Big(2mb^{2m}-(2m-1)b^{2m-2}\Big).\end{aligned}$$

Applying the residue theorem, we can easily prove for any $m\in\mathbb{N}^*$,

$$\fint_{\mathbb{T}} \frac{\overline{\tau}^m}{(1-b\tau)^3}d\tau = \fint_{\mathbb{T}} \frac{\tau^{m+1}}{(\tau-b)^3}d\tau = \frac{m(m+1)}{2}b^{m-1}. \tag{3.7}$$

Thus,

$$\begin{aligned}\fint_{\mathbb{T}} \frac{(1-b\overline{\tau})(v_{1,m}-v_{2,m}\overline{\tau}^{m-1})^2}{(1-b\tau)^3}d\tau &= v_{1,m}v_{2,m}m\Big((m+1)b^m-(m-1)b^{m-2}\Big)-bv_{1,m}^2\\ &+v_{2,m}^2(2m-1)\Big((m-1)b^{2m-3}-mb^{2m-1}\Big).\end{aligned}$$

It follows that

$$\eta_{2,1} = -v_{2,m}^2 mb^{2m} - b^2 v_{1,m}^2 + v_{1,m}v_{2,m}(m+1)b^{m+1}.$$

For the diagonal term we write

$$\eta_{i,i} = \frac{v_{i,m}^2}{b^{i-1}}(m-1)\fint_{\mathbb{T}} \frac{(1-\overline{\tau})(1-\overline{\tau}^{m-1})}{(1-\tau)^2}\overline{\tau}^m d\tau + \frac{v_{i,m}^2}{b^{i-1}}\fint_{\mathbb{T}} \frac{(1-\overline{\tau})(1-\overline{\tau}^{m-1})^2}{(1-\tau)^3}d\tau.$$

By the residue theorem we get

$$\eta_{i,i}=0.$$

Putting together the preceding estimates yields

$$\begin{aligned}\frac{d^2}{dt^2}[I(\Phi_1(t,w))]_{|t=0} &= 2\Big(v_{2,m}^2 mb^{2m}+b^2v_{1,m}^2-v_{1,m}v_{2,m}(m+1)b^{m+1}\Big)\overline{w}^{2m+1}\\ &+ 2(m-1)\big(v_{1,m}^2-v_{2,m}^2\big)\overline{w}\end{aligned}$$

and

$$\frac{d^2}{dt^2}[I(\Phi_2(t,w))]_{|t=0} = 2(m-1)v_{2,m}\Big(v_{1,m}b^{m-2}-\frac{v_{2,m}}{b}\Big)\overline{w}.$$

Combining these estimates with (3.5) and (3.6) we find successively,

$$\begin{aligned}\frac{d^2}{dt^2}[G_1(\lambda_m^\pm,tv_m)]_{|t=0} &= \operatorname{Im}\Big\{w\frac{d^2}{dt^2}I(\Phi_1(t,w))_{|t=0}+2(1-m)v_{1,m}\frac{d}{dt}I(\Phi_1(t,w))_{|t=0}\overline{w}^{m-1}\Big\}\\ &= 2m\big(v_{2,m}b^m-bv_{1,m}\big)^2 e_{2m}\end{aligned}$$

and

$$\begin{aligned}\frac{d^2}{dt^2}[G_2(\lambda_m^\pm,tv_m)]_{|t=0} &= \operatorname{Im}\Big\{bw\frac{d^2}{dt^2}I(\Phi_2(t,w))_{|t=0}+2(1-m)v_{2,m}\frac{d}{dt}I(\Phi_2(t,w))_{|t=0}\overline{w}^{m-1}\Big\}\\ &=0.\end{aligned}$$

This can be written in the form,

$$\frac{d^2}{dt^2}[G(\lambda_m^\pm,tv_m)]_{|t=0} = \begin{pmatrix} 2m(v_{2,m}b^m-bv_{1,m})^2\\ 0\end{pmatrix} e_{2m}.$$

From the structure of the projector Q we get

$$\begin{aligned}\partial_t F_2(\lambda_m^\pm,0) &= \frac{1}{2}Q\partial_{ff}G(\lambda_m^\pm,0)[v_m,v_m]\\ &= \frac{1}{2}Q\frac{d^2}{dt^2}[G(\lambda_m^\pm,tv_m)]_{|t=0}\\ &= 0.\end{aligned}$$

Hence the first point of the Proposition 3.2 is proved.

3.2.2. *Computation of $\partial_\lambda F_2(\lambda_m^\pm,0)$.* From the explicit expression of $\partial_f G(\lambda_m^\pm,0)$ it is easy to verify that

$$\partial_\lambda\partial_f G(\lambda_m^\pm,0)(v_m) = m\begin{pmatrix} v_{1,m}\\ bv_{2,m}\end{pmatrix}e_m. \tag{3.8}$$

Thus we have,

$$\begin{aligned}\partial_\lambda F_2(\lambda_m^\pm,0) &= Q\partial_\lambda\partial_f G(\lambda_m^\pm,0)(v_m)\\ &= m\Big\langle\begin{pmatrix} v_{1,m}\\ bv_{2,m}\end{pmatrix},\widehat{\mathbb{W}}_m\Big\rangle\,\mathbb{W}_m.\end{aligned}$$

Straightforward computations lead to

$$\partial_\lambda F_2(\lambda_m^\pm,0) = \frac{m[(m\lambda_m^\pm-m+1)^2-b^{2m}]}{b^{m-1}[(m\lambda_m^\pm-m+1)^2+b^{2m}]}\begin{pmatrix} m\lambda_m^\pm-m+1\\ -b^m\end{pmatrix}e_m. \tag{3.9}$$

Thus the second point of the Proposition 3.2 follows.

3.2.3. *Computation of $\partial_{\lambda\lambda}F_2(\lambda_m^\pm,0)$.* Using (3.8) and(3.9) we obtain

$$\begin{aligned}(\mathrm{Id}-Q)\partial_\lambda\partial_f G(\lambda_m^\pm,0)v_m &= \frac{2mb(m\lambda_m^\pm-m+1)}{(m\lambda_m^\pm-m+1)^2+b^{2m}}\begin{pmatrix} b^m\\ m\lambda_m^\pm-m+1\end{pmatrix}e_m\\ &= \kappa\begin{pmatrix} b^m\\ m\lambda_m^\pm-m+1\end{pmatrix}e_m\end{aligned}$$

with

$$\kappa \triangleq \frac{2mb(m\lambda_m^\pm-m+1)}{(m\lambda_m^\pm-m+1)^2+b^{2m}}.$$

Then by (2.8) and the expression of $\partial_\lambda\partial_g\varphi(\lambda_m^\pm,0)$ detailed in Proposition 3.1 one gets

$$\begin{aligned}\partial_\lambda\partial_g\varphi(\lambda_m^\pm,0) &= -[\partial_f G(\lambda_m^\pm,0)]^{-1}(\mathrm{Id}-Q)\partial_\lambda\partial_f G(\lambda_m^\pm,0)v_m\\ &= -\kappa[\partial_f G(\lambda_m^\pm,0)]^{-1}\begin{pmatrix} b^m\\ m\lambda_m^\pm-m+1\end{pmatrix}e_m\\ &= \frac{2mb^{1-m}(m\lambda_m^\pm-m+1)^2}{\big(m\lambda_m^\pm-m+1\big)^2+b^{2m}}\begin{pmatrix}1\\0\end{pmatrix}\overline{w}^{m-1}.\end{aligned} \tag{3.10}$$

Consequently,

$$\partial_\lambda\partial_f G(\lambda_m^\pm,0)[\partial_\lambda\partial_g\varphi(\lambda_m^\pm,0)v_m] = \frac{2m^2(m\lambda_m^\pm-m+1)^2}{(m\lambda_m^\pm-m+1)^2+b^{2m}}\begin{pmatrix}1\\0\end{pmatrix}e_m.$$

Straightforward computations lead to

$$Q\partial_\lambda\partial_f G(\lambda_m^\pm,0)[\partial_\lambda\partial_g\varphi(\lambda_m^\pm,0)v_m]=\frac{2m^2b^{1-m}(m\lambda_m^\pm-m+1)^3}{[(m\lambda_m^\pm-m+1)^2+b^{2m}]^2}\begin{pmatrix} m\lambda_m^\pm-m+1\\ -b^m\end{pmatrix}e_m.$$

Finally we obtain the following expression

$$\partial_{\lambda\lambda}F_2(\lambda_m^\pm,0)=\frac{4m^2b^{1-m}(m\lambda_m^\pm-m+1)^3}{[(m\lambda_m^\pm-m+1)^2+b^{2m}]^{\frac{3}{2}}}\mathbb{W}_m.$$

3.2.4. *Computation of* $\partial_{tt}F_2(\lambda_m^\pm,0)$. We mention that most of the computations were done in [18] and so we shall just outline the basic steps. Looking to the formula given in Proposition 3.1 we need first to compute $\frac{d^3}{dt^3}[G(\lambda_m^\pm,tv_m)]_{|t=0}$. From the identity (60) of [18] we recall that

$$\frac{d^3}{dt^3}G_j(\lambda_m^\pm,tv_m)_{|t=0}=\operatorname{Im}\left\{b^{j-1}w\frac{d^3}{dt^3}[I(\Phi_j(t,w))]_{|t=0}+3(1-m)v_{j,m}\overline{w}^{m-1}[\frac{d^2}{dt^2}(I(\Phi_j(t,w)))]_{|t=0}\right\}.$$

It is also shown in [18, p. 835] that

$$\frac{d^3}{dt^3}[I_i(\Phi_j(t,w))]_{|t=0}=-6\fint_{\mathbb{T}}\frac{[\overline{B}A-\overline{A}B]}{A^4}B[AC-b^{i-1}B]d\tau.$$

One can find real numbers $\widehat{\eta}_{i,j}$ such that

$$\frac{1}{6}\frac{d^3}{dt^3}[I_i(\Phi_j(t,w))]_{|t=0}=(m-1)J_{i,j}\overline{w}^{m+1}+b^{i-1}K_{i,j}\overline{w}^{m+1}+\widehat{\eta}_{i,j}\overline{w}^{3m+1}$$

where

$$J_{i,j}=v_{i,m}\fint_{\mathbb{T}}\frac{(v_{j,m}-v_{i,m}\overline{\tau}^{m-1})(v_{j,m}-v_{i,m}\tau^{m-1})}{(b^{j-1}-b^{i-1}\tau)^2}\overline{\tau}^m d\tau$$

and

$$K_{i,j}=\fint_{\mathbb{T}}\frac{(v_{j,m}-v_{i,m}\overline{\tau}^{m-1})^2(v_{j,m}-v_{i,m}\tau^{m-1})}{(b^{j-1}-b^{i-1}\tau)^3}d\tau.$$

To start, we compute $J_{i,j}$. The same proof of (61) of [18] gives

$$J_{1,2}=0.$$

For $J_{2,1}$ we use (3.4),

$$\begin{aligned}J_{2,1}&=v_{2,m}\fint_{\mathbb{T}}\frac{(v_{1,m}-v_{2,m}\overline{\tau}^{m-1})(v_{1,m}-v_{2,m}\tau^{m-1})}{(1-b\tau)^2}\overline{\tau}^m d\tau\\ &=v_{2,m}[v_{1,m}^2+v_{2,m}^2]mb^{m-1}+v_{1,m}v_{2,m}^2[(1-2m)b^{2m-2}-1].\end{aligned}$$

As to $J_{i,i}$ we easily get

$$\begin{aligned}J_{i,i}&=\frac{v_{i,m}^3}{b^{2(i-1)}}\fint_{\mathbb{T}}\frac{(1-\overline{\tau}^{m-1})(1-\tau^{m-1})}{(1-\tau)^2}\overline{\tau}^m d\tau\\ &=0.\end{aligned}$$

For $K_{1,2}$ we write

$$\begin{aligned}
K_{1,2} &= \fint_{\mathbb{T}} \frac{v_{2,m}^3 + v_{1,m}^2 v_{2,m}\overline{\tau}^{2m-2} - 2v_{2,m}^2 v_{1,m}\overline{\tau}^{m-1} - v_{1,m}^3\overline{\tau}^{m-1} + 2v_{1,m}^2 v_{2,m}}{(b-\tau)^3} d\tau \\
&- v_{1,m}v_{2,m}^2 \fint_{\mathbb{T}} \frac{\tau^{m-1}}{(b-\tau)^3} d\tau.
\end{aligned}$$

Using the residue theorem at ∞, we can see that all the terms vanish except the last one that can computed also with the residue theorem.

$$K_{1,2} = \frac{v_{1,m}v_{2,m}^2(m-1)(m-2)}{2} b^{m-3}.$$

For $K_{2,1}$ given by

$$K_{2,1} = \fint_{\mathbb{T}} \frac{v_{1,m}^3 - v_{1,m}^2 v_{2,m}\tau^{m-1} + 2v_{1,m}v_{2,m}^2 + v_{1,m}v_{2,m}^2\overline{\tau}^{2m-2} - v_{2,m}^3\overline{\tau}^{m-1} - 2v_{1,m}^2 v_{2,m}\overline{\tau}^{m-1}}{(1-b\tau)^3} d\tau.$$

we may use the residue theorem combined with (3.7)

$$\begin{aligned}
K_{2,1} &= \fint_{\mathbb{T}} \frac{v_{1,m}v_{2,m}^2\overline{\tau}^{2m-2} - v_{2,m}^3\overline{\tau}^{m-1} - 2v_{1,m}^2 v_{2,m}\overline{\tau}^{m-1}}{(1-b\tau)^3} d\tau \\
&= (2v_{1,m}^2 + v_{2,m}^2)v_{2,m}\frac{m(1-m)}{2} b^{m-2} + v_{1,m}v_{2,m}^2(m-1)(2m-1)b^{2m-3}.
\end{aligned}$$

As to the diagonal terms $K_{i,i}$ we use residue theorem leading to

$$\begin{aligned}
K_{i,i} &= \frac{v_{i,m}^3}{b^{3(i-1)}} \fint_{\mathbb{T}} \frac{(1-\overline{\tau}^{m-1})^2(1-\tau^{m-1})}{(1-\tau)^3} d\tau \\
&= \frac{v_{i,m}^3}{b^{3(i-1)}} \frac{(m-1)(m-2)}{2}
\end{aligned}$$

Summing up we find

$$\begin{aligned}
\frac{1}{6}\frac{d^3}{dt^3}[I(\Phi_1(t,w))]_{|t=0} &= (m-1)\Big(v_{2,m}^2 v_{1,m} - v_{2,m}^3 \frac{m}{2} b^{m-1} + v_{1,m}^3 \frac{m-2}{2}\Big)\overline{w}^{m+1} \\
&+ (\widehat{\eta}_{1,1} - \widehat{\eta}_{2,1})\overline{w}^{3m+1}
\end{aligned}$$

and

$$\begin{aligned}
\frac{1}{6}\frac{d^3}{dt^3}[I(\Phi_2(t,w))]_{|t=0} &= \frac{v_{2,m}^2(m-1)(m-2)}{2}\Big(v_{1,m}b^{m-3} - \frac{v_{2,m}}{b^2}\Big)\overline{w}^{m+1} \\
&+ (\widehat{\eta}_{1,2} - \widehat{\eta}_{2,2})\overline{w}^{3m+1}.
\end{aligned}$$

This leads to

$$\begin{aligned}
\frac{d^3}{dt^3}[G_1(\lambda_m^{\pm}, tv_m)]_{|t=0} &= \operatorname{Im}\Big\{6(m-1)\Big(v_{2,m}^2 v_{1,m} - v_{2,m}^3\frac{m}{2}b^{m-1} + v_{1,m}^3\frac{m-2}{2}\Big)\overline{w}^m. \\
&+ 6(\widehat{\eta}_{1,1} - \widehat{\eta}_{2,1})\overline{w}^{3m} + 6(1-m)v_{1,m}(m-1)\Big(v_{1,m}^2 - v_{2,m}^2\Big)\overline{w}^m \\
&+ 6(1-m)v_{1,m}\Big(v_{2,m}^2 m b^{2m} + b^2 v_{1,m}^2 - v_{1,m}v_{2,m}(m+1)b^{m+1}\Big)\overline{w}^{3m}\Big\} \\
&= 3m(m-1)\Big(2v_{2,m}^2 v_{1,m} - v_{2,m}^3 b^{m-1} - v_{1,m}^3\Big)e_m + \gamma_1 e_{3m}
\end{aligned}$$

and

$$\begin{aligned}\frac{d^3}{dt^3}[G_2(\lambda_m^\pm,0)]_{|t=0} &= \mathrm{Im}\Big\{3v_{2,m}^2(m-1)(m-2)\Big(v_{1,m}b^{m-2}-\frac{v_{2,m}}{b}\Big)\overline{w}^m + b(\widehat{\eta}_{1,2}-\widehat{\eta}_{2,2})\overline{w}^{3m}\\ &\quad +6(1-m)v_{2,m}^2(m-1)\Big(v_{1,m}b^{m-2}-\frac{v_{2,m}}{b}\Big)\overline{w}^m\Big\}\\ &= 3v_{2,m}^2(m-1)m\Big(\frac{v_{2,m}}{b}-v_{1,m}b^{m-2}\Big)e_m+\gamma_2 e_{3m}\end{aligned}$$

with $\gamma_j\in\mathbb{R}$. In summary,

$$\frac{d^3}{dt^3}[G(\lambda_m^\pm,0)]_{|t=0} = 3m(m-1)\left(\begin{array}{c} 2v_{2,m}^2v_{1,m}-v_{2,m}^3b^{m-1}-v_{1,m}^3\\ v_{2,m}^2\Big(\frac{v_{2,m}}{b}-v_{1,m}b^{m-2}\Big)\end{array}\right)e_m+\left(\begin{array}{c}\gamma_1\\ \gamma_2\end{array}\right)e_{3m}.$$

Using the structure of the projector Q we deduce after algebraic cancellations

$$\frac{1}{3}\frac{d^3}{dt^3}[QG(\lambda_m^\pm,tv_m)]_{|t=0} = -m(m-1)b^{3-3m}\frac{(b^{2m-2}-(m\lambda_m^\pm-m+1)^2)^2}{([m\lambda_m^\pm-m+1]^2+b^{2m})^{\frac12}}\mathbb{W}_m.$$

Now we shall compute the term,

$$Q\partial_{ff}G(\lambda_m^\pm,0)[v_m,\widehat{v}_m] = Q\partial_t\partial_s[G(\lambda_m^\pm,tv_m+s\widehat{v}_m)]_{|t=0,s=0}.$$

To find an expression of $\widehat{v}_m$, we recall that

$$\frac{d^2}{dt^2}[G(\lambda_m^\pm,tv_m)]_{|t=0} = \left(\begin{array}{c}\widehat{\alpha}\\ 0\end{array}\right)e_{2m}\quad\text{with}\quad \widehat{\alpha}=2m(v_{2,m}b^m-bv_{1,m})^2.$$

Thus,

$$\begin{aligned}\widehat{v}_m &= -[\partial_f G(\lambda_m^\pm,0)]^{-1}\left(\begin{array}{c}\widehat{\alpha}\\ 0\end{array}\right)e_{2m}\\ &= -M_{2m}^{-1}\left(\begin{array}{c}\widehat{\alpha}\\ 0\end{array}\right)\overline{w}^{2m-1}\\ &= \left(\begin{array}{c}\widehat{v}_{1,m}\\ \widehat{v}_{2,m}\end{array}\right)\overline{w}^{2m-1}\end{aligned}\tag{3.11}$$

with

$$\widehat{v}_{1,m} = -\frac{b\widehat{\alpha}(2m\lambda_m^\pm-2m+1)}{\det(M_{2m}(\lambda_m^\pm))}\quad\text{and}\quad \widehat{v}_{2,m}=-\frac{b^{2m}\widehat{\alpha}}{\det(M_{2m}(\lambda_m^\pm))}.$$

Finally we can write,

$$\widehat{v}_m=\widetilde{\beta}_m\tilde{v}_m$$

with

$$\widetilde{\beta}_m=-\frac{b\widehat{\alpha}}{\det(M_{2m}(\lambda_m^\pm))}\tag{3.12}$$

$$\tilde{v}_m=\left(\begin{array}{c}2m\lambda_m^\pm-2m+1\\ b^{2m-1}\end{array}\right)\overline{w}^{2m-1}\triangleq\left(\begin{array}{c}\beta_1\\ \beta_2\end{array}\right)\overline{w}^{2m-1}.$$

It follows that

$$Q\partial_t\partial_s[G(\lambda_m^\pm,tv_m+s\widehat{v}_m)]_{|t=0,s=0}=\tilde{\beta}_mQ\partial_t\partial_s[G(\lambda_m^\pm,tv_m+s\tilde{v}_m)]_{|t=0,s=0}.$$

We shall introduce the functions

$$\varphi_j(t,s,w) = b^{j-1}w + tv_{j,m}w^{1-m} + s\beta_j w^{1-2m}$$

and hence

$$\begin{aligned} G_j(t,s,w) &\triangleq G_j(\lambda, tv_m + s\tilde{v}_m) \\ &= \operatorname{Im}\left\{\left[(1-\lambda_m^{\pm})\overline{\varphi_j(t,s,w)} + I(\varphi_j(t,s,w))\right] w\partial_w\varphi_j(t,s,w)\right\}. \end{aligned}$$

The following equality can be easily check :

$$\begin{aligned} \partial_t\partial_s[G_j(t,s,w)]_{|t=0,s=0} &= \operatorname{Im}\Big\{(1-\lambda_m^{\pm})\beta_j v_{j,m}\Big((1-m)w^m + (1-2m)\overline{w}^m\Big)\Big\} \\ &+ \operatorname{Im}\Big\{wb^{j-1}\frac{d^2}{dtds}[I(\varphi_j(t,s,w))]_{|t=0,s=0}\Big\} \\ &+ \operatorname{Im}\Big\{\beta_j(1-2m)\partial_t[I(\varphi_j(t,s,w))]_{|t=0,s=0}\overline{w}^{2m-1}\Big\} \\ &+ \operatorname{Im}\Big\{(1-m)v_{j,m}\overline{w}^{m-1}\partial_s[I(\varphi_j(t,s,w))]_{|t=0,s=0}\Big\}. \end{aligned}$$

We write

$$\begin{aligned} I_i(\varphi_j(t,s,w)) &= \fint_{\mathbb{T}} \frac{\overline{\varphi_j(t,s,w)} - \overline{\varphi_i(t,s,\tau)}}{\varphi_j(t,s,w) - \varphi_i(t,s,\tau)}\Big(b^{i-1} + t(1-m)v_{i,m}\overline{\tau}^m + s\beta_i(1-2m)\overline{\tau}^{2m}\Big)d\tau \\ &= \fint_{\mathbb{T}} \frac{\overline{A} + t\overline{B} + s\overline{C}}{A + tB + sC}\big(b^{i-1} + t\,D + s\,E\big)d\tau \end{aligned}$$

with

$$A = b^{j-1}w - b^{i-1}\tau,\ B = v_{j,m}\overline{w}^{m-1} - v_{i,m}\overline{\tau}^{m-1},\ C = \beta_j\overline{w}^{2m-1} - \beta_i\overline{\tau}^{2m-1}$$
$$D = (1-m)v_{i,m}\overline{\tau}^m \quad \text{and} \quad E = (1-2m)\beta_i\overline{\tau}^{2m}.$$

Straightforward computations lead to

$$\begin{aligned} \partial_t[I_i(\varphi_j(t,s,w)]_{|t=0,s=0} &= b^{i-1}\fint_{\mathbb{T}} \frac{A\overline{B} - \overline{A}B}{A^2}d\tau + \fint_{\mathbb{T}} \frac{\overline{A}D}{A}d\tau \\ &= \tilde{J}_{i,j}w^{m-1} + \theta_{i,j}\overline{w}^{m+1} \end{aligned}$$

with $\theta_{i,j} \in \mathbb{R}$ and

$$\tilde{J}_{i,j} = b^{i-1}\fint_{\mathbb{T}} \frac{(v_{j,m} - v_{i,m}\tau^{m-1})}{(b^{j-1} - b^{i-1}\tau)}d\tau.$$

For the diagonal tens one has

$$\begin{aligned} \tilde{J}_{i,i} &= v_{i,m}\fint_{\mathbb{T}} \frac{(1-\tau^{m-1})}{(1-\tau)}d\tau \\ &= 0 \end{aligned}$$

On the other hand

$$\begin{aligned} \tilde{J}_{1,2} &= \fint_{\mathbb{T}} \frac{(v_{2,m} - v_{1,m}\tau^{m-1})}{(b-\tau)}d\tau \\ &= v_{1,m}b^{m-1} - v_{2,m}. \end{aligned}$$

Using again the residue theorem it is easy to see that,

$$\tilde{J}_{2,1} = b \fint_{\mathbb{T}} \frac{(v_{1,m} - v_{2,m}\tau^{m-1})}{(1-b\tau)} d\tau = 0.$$

Summing up we obtain,

$$\partial_t[I(\varphi_1(t,s,w))]_{|t=0,s=0} = (\theta_{1,1} - \theta_{2,1})\overline{w}^{m+1}$$

and

$$\partial_t[I(\varphi_2(t,s,w))]_{|t=0,s=0} = (\theta_{1,2} - \theta_{2,2})\overline{w}^{m+1} + (v_{1,m}b^{m-1} - v_{2,m})w^{m-1}.$$

For the derivative with respect to s we may write

$$\begin{aligned}
\partial_s[I_i(\varphi_j(t,s,w))]_{|t=0,s=0} &= b^{i-1} \fint_{\mathbb{T}} \frac{A\overline{C} - \overline{A}C}{A^2} d\tau + \fint_{\mathbb{T}} \frac{\overline{A}E}{A} d\tau \\
&= \widehat{J}_{i,j} w^{2m-1} + \widehat{\theta}_{i,j}\overline{w}^{2m+1}
\end{aligned}$$

where

$$\widehat{J}_{i,j} = b^{i-1} \fint_{\mathbb{T}} \frac{(\beta_j - \beta_i\tau^{2m-1})}{(b^{j-1} - b^{i-1}\tau)} d\tau$$

and

$$\widehat{\theta}_{i,j} = \fint_{\mathbb{T}} \frac{(b^{j-1} - b^{i-1}\overline{\tau})}{(b^{j-1} - b^{i-1}\tau)}(1-2m)\beta_i\overline{\tau}^{2m} d\tau - b^{i-1} \fint_{\mathbb{T}} \frac{(b^{j-1} - b^{i-1}\overline{\tau})(\beta_j - \beta_i\overline{\tau}^{2m-1})}{(b^{j-1} - b^{i-1}\tau)^2} d\tau.$$

It is easy to check that $\widehat{\theta}_{i,j} \in \mathbb{R}$, $\forall i,j \in \{1,2\}$. Now, we get

$$\widehat{J}_{i,i} = \beta_i \fint_{\mathbb{T}} \frac{(1-\tau^{2m-1})}{(1-\tau)} d\tau = 0.$$

Using the residue theorem we find

$$\widehat{J}_{1,2} = \fint_{\mathbb{T}} \frac{(\beta_2 - \beta_1\tau^{2m-1})}{(b-\tau)} d\tau = -\beta_2 + \beta_1 b^{2m-1}.$$

and

$$\widehat{J}_{2,1} = b \fint_{\mathbb{T}} \frac{(\beta_1 - \beta_2\tau^{2m-1})}{(1-b\tau)} d\tau = 0.$$

To summarize,

$$\partial_s[I(\varphi_1(t,s,w))]_{|t=0,s=0} = (\widehat{\theta}_{1,1} - \widehat{\theta}_{2,1})\overline{w}^{2m+1}$$

and

$$\partial_s[I(\varphi_2(t,s,w))]_{|t=0,s=0} = (\beta_1 b^{2m-1} - \beta_2)w^{2m-1} + (\widehat{\theta}_{1,2} - \widehat{\theta}_{2,2})\overline{w}^{2m+1}.$$

Now we shall move to the second derivative with respect to t and s,

$$\begin{aligned}
\frac{d^2}{dsdt}[I_i(\varphi_j(t,s,w))]_{|t=0,s=0} = &-b^{i-1} \fint_{\mathbb{T}} \frac{\overline{B}C}{A^2} d\tau + \fint_{\mathbb{T}} \frac{\overline{B}E}{A} d\tau - b^{i-1} \fint_{\mathbb{T}} \frac{\overline{C}B}{A^2} d\tau + \fint_{\mathbb{T}} \frac{\overline{C}D}{A} d\tau \\
&+ 2b^{i-1} \fint_{\mathbb{T}} \frac{BC\overline{A}}{A^3} d\tau - \fint_{\mathbb{T}} \frac{BE\overline{A}}{A^2} d\tau - \fint_{\mathbb{T}} \frac{DC\overline{A}}{A^2} d\tau.
\end{aligned}$$

By homogeneity, there exist $\varepsilon_{i,j} \in \mathbb{R}$ such that,

$$\frac{d^2}{dsdt}[I_i(\varphi_j(t,s,w))]_{|t=0,s=0}=\varepsilon_{i,j}\overline{w}^{3m+1}-b^{i-1}I_1^{i,j}w^{m-1}-b^{i-1}I_2^{i,j}\overline{w}^{m+1}+I_3^{i,j}\overline{w}^{m+1}+I_4^{i,j}w^{m-1}$$

with

$$\begin{aligned}
I_1^{i,j}&=\fint_{\mathbb{T}}\frac{(v_{j,m}-v_{i,m}\overline{\tau}^{m-1})(\beta_j-\beta_i\tau^{2m-1})}{(b^{j-1}-b^{i-1}\tau)^2}d\tau,\\
I_2^{i,j}&=\fint_{\mathbb{T}}\frac{(v_{j,m}-v_{i,m}\tau^{m-1})(\beta_j-\beta_i\overline{\tau}^{2m-1})}{(b^{j-1}-b^{i-1}\tau)^2}d\tau,\\
I_3^{i,j}&=(1-2m)\beta_i\fint_{\mathbb{T}}\frac{(v_{j,m}-v_{i,m}\tau^{m-1})}{(b^{j-1}-b^{i-1}\tau)}\overline{\tau}^{2m}d\tau
\end{aligned}$$

and

$$I_4^{i,j}=(1-m)v_{i,m}\fint_{\mathbb{T}}\frac{(\beta_j-\beta_i\tau^{2m-1})}{(b^{j-1}-b^{i-1}\tau)}\overline{\tau}^{m}d\tau.$$

We intend to compute all these terms. For the diagonal terms we write

$$\begin{aligned}
I_1^{i,i}&=\frac{v_{i,m}\beta_i}{b^{2(i-1)}}\fint_{\mathbb{T}}\frac{(1-\overline{\tau}^{m-1})(1-\tau^{2m-1})}{(1-\tau)^2}d\tau\\
&=(1-m)\frac{v_{i,m}\beta_i}{b^{2(i-1)}}.
\end{aligned}$$

As to the term $I_1^{1,2}$ we have

$$I_1^{1,2}=\fint_{\mathbb{T}}\frac{v_{2,m}\beta_2}{(b-\tau)^2}d\tau-\fint_{\mathbb{T}}\frac{v_{1,m}\beta_2\overline{\tau}^{m-1}}{(b-\tau)^2}d\tau-\fint_{\mathbb{T}}\frac{v_{2,m}\beta_1\tau^{2m-1}}{(b-\tau)^2}d\tau+\fint_{\mathbb{T}}\frac{v_{1,m}\beta_1\tau^{m})}{(b-\tau)^2}d\tau.$$

By the residue theorem we get

$$I_1^{1,2}=v_{2,m}\beta_1(1-2m)b^{2m-2}+m\beta_1v_{1,m}b^{m-1}.$$

Now we move to $I_1^{2,1}$. Residue theorem combined with (3.4) implies

$$\begin{aligned}
I_1^{2,1}&=\fint_{\mathbb{T}}\frac{v_{1,m}\beta_1}{(1-b\tau)^2}d\tau+\fint_{\mathbb{T}}\frac{v_{2,m}\beta_2\tau^{m}}{(1-b\tau)^2}d\tau-\fint_{\mathbb{T}}\frac{v_{1,m}\beta_2\tau^{2m-1}}{(1-b\tau)^2}d\tau-\fint_{\mathbb{T}}\frac{v_{2,m}\beta_1\overline{\tau}^{m-1}}{(1-b\tau)^2}d\tau\\
&=-v_{2,m}\beta_1(m-1)b^{m-2}.
\end{aligned}$$

Moreover,

$$\begin{aligned}
I_2^{i,i}&=\frac{v_{i,m}\beta_i}{b^{2(i-1)}}\fint_{\mathbb{T}}\frac{(1-\tau^{m-1})(1-\overline{\tau}^{2m-1})}{(1-\tau)^2}d\tau\\
&=(1-m)\frac{v_{i,m}\beta_i}{b^{2(i-1)}}.
\end{aligned}$$

For $I_2^{i,j}$ we use the change of variable $\tau \to \overline{\tau}$

$$I_2^{i,j} = \fint_{\mathbb{T}} \frac{(v_{j,m} - v_{i,m}\tau^{m-1})(\beta_j - \beta_i \overline{\tau}^{2m-1})}{(b^{j-1} - b^{i-1}\tau)^2} d\tau$$
$$= \fint_{\mathbb{T}} \frac{(v_{j,m} - v_{i,m}\overline{\tau}^{m-1})(\beta_j - \beta_i \tau^{2m-1})}{(b^{i-1} - b^{j-1}\tau)^2} d\tau.$$

Therefore

$$I_2^{1,2} = \fint_{\mathbb{T}} \frac{(v_{2,m} - v_{1,m}\overline{\tau}^{m-1})(\beta_2 - \beta_1 \tau^{2m-1})}{(1 - b\tau)^2} d\tau.$$

Similarly to $I_1^{2,1}$ we find

$$I_2^{1,2} = -v_{1,m}\beta_2(m-1)b^{m-2}.$$

For the term $I_2^{2,1}$ we write

$$I_2^{2,1} = \fint_{\mathbb{T}} \frac{(v_{1,m} - v_{2,m}\overline{\tau}^{m-1})(\beta_1 - \beta_2 \tau^{2m-1})}{(b - \tau)^2} d\tau.$$

The same computations for $I_1^{1,2}$ yield

$$I_2^{2,1} = v_{1,m}\beta_2(1-2m)b^{2m-2} + m\beta_2 v_{2,m} b^{m-1}.$$

For the diagonal term $I_3^{i,i}$ we easily get

$$I_3^{i,i} = (1-2m)\frac{v_{i,m}\beta_i}{b^{i-1}} \fint_{\mathbb{T}} \frac{(1 - \tau^{m-1})}{(1-\tau)} \overline{\tau}^{2m} d\tau = 0.$$

Moreover,

$$I_3^{1,2} = (1-2m)\beta_i \fint_{\mathbb{T}} \frac{(v_{2,m} - v_{1,m}\tau^{m-1})}{(b-\tau)} \overline{\tau}^{2m} d\tau = 0.$$

On the other hand, using (3.3) we find

$$\begin{aligned} I_3^{2,1} &= (1-2m)\beta_2 \fint_{\mathbb{T}} \frac{(v_{1,m} - v_{2,m}\tau^{m-1})}{(1-b\tau)} \overline{\tau}^{2m} d\tau \\ &= (1-2m)\beta_2\big(v_{1,m}b^{2m-1} - v_{2,m}b^m\big). \end{aligned}$$

Now we move to the last terms $I_4^{i,j}$. Concerning the diagonal terms, we may write

$$\begin{aligned} I_4^{i,i} &= (1-m)\frac{v_{i,m}\beta_i}{b^{i-1}} \fint_{\mathbb{T}} \frac{(1-\tau^{2m-1})}{(1-\tau)} \overline{\tau}^m d\tau \\ &= (1-m)\frac{v_{i,m}\beta_i}{b^{i-1}}. \end{aligned}$$

For $I_4^{1,2}$ we obtain according to the residue theorem

$$\begin{aligned} I_4^{1,2} &= (1-m)v_{1,m} \fint_{\mathbb{T}} \frac{\beta_2 \overline{\tau}^m}{(b-\tau)} d\tau - (1-m)v_{1,m} \fint_{\mathbb{T}} \frac{\beta_1 \tau^{m-1}}{(b-\tau)} d\tau \\ &= (1-m)v_{1,m}\beta_1 b^{m-1}. \end{aligned}$$

For the last term, we use (3.3) in order to get

$$\begin{aligned} I_4^{2,1} &= (1-m)v_{2,m}\fint_{\mathbb{T}}\frac{(\beta_1-\beta_2\tau^{2m-1})}{(1-b\tau)}\overline{\tau}^m d\tau \\ &= (1-m)v_{2,m}\beta_1 b^{m-1}. \end{aligned}$$

Putting together the preceding identities we deduce

$$\frac{d^2}{dsdt}[I(\varphi_1(t,s,w))]_{|t=0,s=0} = (\varepsilon_{1,1}-\varepsilon_{2,1})\overline{w}^{3m+1} + (m-1)\big(v_{1,m}\beta_1-\beta_2 v_{2,m}b^m\big)\overline{w}^{m+1}$$

and

$$\begin{aligned} \frac{d^2}{dsdt}[I(\varphi_2(t,s,w))]_{|t=0,s=0} &= (\varepsilon_{1,2}-\varepsilon_{2,2})\overline{w}^{3m+1} + (1-2m)\beta_1\big(v_{1,m}b^{m-1}-v_{2,m}b^{2m-2}\big)w^{m-1} \\ &+ (m-1)\beta_2\big(v_{1,m}b^{m-2}-\frac{v_{2m}}{b}\big)\overline{w}^{m+1}. \end{aligned}$$

Finally we get,

$$\begin{aligned} \frac{d^2}{dtds}[G_1(t,s,w)]_{|t=0,s=0} &= \operatorname{Im}\Big\{(\varepsilon_{1,1}-\varepsilon_{2,1})\overline{w}^{3m} + (m-1)[v_{1,m}\beta_1-\beta_2 v_{2,m}b^m]\overline{w}^m \\ &+ \beta_1(1-2m)(\theta_{1,1}-\theta_{2,1})\overline{w}^{3m}\Big\} \\ &+ \operatorname{Im}\Big\{(\lambda_m^{\pm}-1)m\beta_1 v_{1,m}\overline{w}^m] + (1-m)v_{1,m}(\widehat{\theta}_{1,1}-\widehat{\theta}_{2,1})\overline{w}^{3m}\Big\} \\ &= \big((m\lambda_m^{\pm}-1)v_{1,m}\beta_1 + (1-m)\beta_2 v_{2,m}b^m\big)\, e_m + \tilde{\gamma}_1 e_{3m} \\ &= \Big((m\lambda_m^{\pm}-1)(m\lambda_m^{\pm}-m+1)(2m\lambda_m^{\pm}-2m+1)b^{1-m} + (1-m)b^{3m-1}\Big)e_m \\ &+ \tilde{\gamma}_1 e_{3m} \end{aligned}$$

and

$$\begin{aligned} \frac{d^2}{dtds}[G_2(t,s,w)]_{|t=0,s=0} &= \operatorname{Im}\Big\{b(\varepsilon_{1,2}-\varepsilon_{2,2})\overline{w}^{3m} + (1-2m)\beta_1[v_{1,m}b^m - v_{2,m}b^{2m-1}]w^m\Big\} \\ &+ \operatorname{Im}\Big\{(m-1)\beta_2[v_{1,m}b^{m-1}-v_{2m}]\overline{w}^m + \beta_2(1-2m)(\theta_{1,2}-\theta_{2,2})\overline{w}^{3m} \\ &+ (1-\lambda_m^{\pm})\beta_2 v_{2,m}[(1-m)w^m + (1-2m)\overline{w}^m] + (1-m)v_{2,m}(\beta_1 b^{2m-1}-\beta_2)w^m\Big\} \\ &+ \operatorname{Im}\Big\{\beta_2(1-2m)(v_{1,m}b^{m-1}-v_{2,m})\overline{w}^m + (1-m)v_{2,m}(\widehat{\theta}_{1,2}-\widehat{\theta}_{2,2})\overline{w}^{3m}\Big\} \\ &= \Big([(m\lambda_m^{\pm}-m+1)\beta_2 - m\beta_1 b^{2m-1}]v_{2,m} + v_{1,m}b^{m-1}[b\beta_1(2m-1)-m\beta_2]\Big)e_m \\ &+ \tilde{\gamma}_2 e_{3m} \\ &= \Big(\big((1-m)(m\lambda_m^{\pm}-m+1) - m(2\lambda_m^{\pm}m-2m+1)\big)b^{2m-1} \\ &+ (m\lambda_m^{\pm}-m+1)b(2m\lambda_m^{\pm}-2m+1)(2m-1)\Big)e_m + \tilde{\gamma}_2 e_{3m}. \end{aligned}$$

Using the definition of the projector Q, we deduce after some computations

$$Q\frac{d^2}{dtds}G(t,s,w)_{|t=0,s=0} = \mathcal{K}_m\,\mathbb{W}_m$$

with

$$\mathcal{K}_m \triangleq (2\lambda_m^{\pm} m - 2m+1)\frac{b^{1-m}(m\lambda_m^{\pm}-1)(m\lambda_m^{\pm}-m+1)^2+(1-2m)(m\lambda_m^{\pm}-m+1)b^{m+1}+mb^{3m-1}}{[(m\lambda_m^{\pm}-m+1)^2+b^{2m}]^{\frac{1}{2}}}.$$

Eventually, we find

$$Q\partial_{ff}G(\lambda_m^{\pm},0)[v_m,\partial_\lambda\partial_g\varphi(\lambda_m^{\pm},0)v_m]=\widetilde{\beta}_m\mathcal{K}_m\mathbb{W}_m$$

where $\widetilde{\beta}_m$ was defined in (3.12). This achieves the proof of Proposition 3.2-(4).

3.2.5. *Computation of* $\partial_\lambda\partial_t F_2(\lambda_m^{\pm},0)$. Now we shall prove the last point of Proposition 3.2. Recall from Proposition 3.1 that

$$\begin{aligned}\partial_\lambda\partial_t F_2(\lambda_m^{\pm},0)&=\frac{1}{2}Q\partial_\lambda\partial_{ff}G(\lambda_m^{\pm},0)[v_m,v_m]+\frac{1}{2}Q\partial_\lambda\partial_f G(\lambda_m^{\pm},0)(\widehat{v}_m)\\&\quad+Q\partial_{ff}G(\lambda_m^{\pm},0)[v_m,\partial_\lambda\partial_g\varphi(\lambda_m^{\pm},0)v_m].\end{aligned}$$

The first term vanishes since

$$\begin{aligned}\partial_\lambda\partial_{ff}G(\lambda_m^{\pm},0)[v_m,v_m]&=\frac{d^2}{dt^2}[\partial_\lambda G_j(\lambda_m^{\pm},tv_m)]_{|\lambda=\lambda_m^{\pm},t=0}\\&=-\frac{d^2}{dt^2}_{|t=0}\operatorname{Im}\left\{w\overline{\Phi_j(t,w)}\Phi_j'(t,w)\right\}\\&=0.\end{aligned}$$

For the second term we combine (2.4) with (3.11)

$$\partial_\lambda\partial_f G(\lambda_m^{\pm},0)(\widehat{v}_m)=2m\begin{pmatrix}\widehat{v}_{1,m}\\ b\widehat{v}_{2,m}\end{pmatrix}e_{2m}.$$

Consequently we deduce that

$$Q\partial_\lambda\partial_f G(\lambda_m^{\pm},0)(\widehat{v}_m)=0.$$

Hence we find

$$\partial_\lambda\partial_t F_2(\lambda_m^{\pm},0)=Q\partial_{ff}G(\lambda_m^{\pm},0)[v_m,\partial_\lambda\partial_g\varphi(\lambda_m^{\pm},0)v_m].$$

Now we want to compute

$$Q\partial_{ff}G(\lambda_m^{\pm},0)[v_m,\partial_\lambda\partial_g\varphi(\lambda_m^{\pm},0)v_m]=Q\partial_t\partial_s[G(\lambda_m^{\pm},tv_m+s\partial_\lambda\partial_g\varphi(\lambda_m^{\pm},0)v_m)]_{|t=0,s=0}.$$

This was done in [18, Lemma 2-(ii)] combined with (3.10) and (3.11). We obtain

$$Q\partial_{ff}G(\lambda_m^{\pm},0)[v_m,\partial_\lambda\partial_g\varphi(\lambda_m^{\pm},0)v_m]=0$$

and this completes the proof of the desired result.

4. Proof of the main theorem

This section is devoted to the proof of Theorem 1.2. To begin, we choose a small neighborhood of $(\lambda_m^+, 0)$ in the strong topology of $\mathbb{R} \times X_m$ such that the equation

$$F_1(\lambda, tv_m, k) = 0, \quad k \in \mathcal{X}_m$$

admits locally a unique surface of solutions parametrized by

$$(\lambda, t) \in (\lambda_m^+ - \epsilon_0, \lambda_m^+ + \epsilon_0) \times (-\epsilon_0, \epsilon_0) \mapsto k = \varphi(\lambda, tv_m) \in \mathcal{X}_m,$$

with $\epsilon_0 > 0$ and φ being a C^1 function and actually it is of class C^k for any $k \in \mathbb{N}$. This follows from the fact that the functionals defining the V-states are better than C^1 and it could be proved that they are in fact of class C^k. Note also that

$$\varphi(\lambda_m^+, 0) = 0.$$

For more details we refer to the subsection 2.3. We recall from that subsection that the V-states equation is equivalent to,

$$F_2(\lambda, t) = 0 \tag{4.1}$$

with (λ, t) being in the neighborhood of $(\lambda_m^+, 0)$ in $\mathbb{R}^2$ and $F_2 : \mathbb{R}^2 \to \langle \mathbb{W}_m \rangle$. We intend to prove the following assertion: there exists $b_m \in (0, b_m^*)$ such that for any $b \in (b_m, b_m^*)$ there exists $\varepsilon > 0$ such that the set

$$\mathcal{E}_b \triangleq \big\{(\lambda, t) \in (\lambda_m^+ - \varepsilon, \lambda_m^+ + \varepsilon) \times (-\varepsilon, \varepsilon),\ F_2(\lambda, t) = 0\big\} \tag{4.2}$$

is a C^1-Jordan curve in the complex plane. Taylor expansion of F_2 around the point $(\lambda_m^+, 0)$ at the order two is given by,

$$\begin{aligned}
F_2(\lambda, t) &= \partial_\lambda F_2(\lambda_m^+, 0)(\lambda - \lambda_m^+) + \partial_t F_2(\lambda_m^+, 0) t \\
&+ \frac{1}{2}\partial_{\lambda\lambda} F_2(\lambda_m^+, 0)(\lambda - \lambda_m^+)^2 + \frac{1}{2}\partial_{tt} F_2(\lambda_m^+, 0) t^2 \\
&+ ((\lambda - \lambda_m^+)^2 + t^2)\epsilon(\lambda, t)
\end{aligned}$$

where

$$\lim_{(\lambda,t)\to(\lambda_m^+,0)} \epsilon(\lambda, t) = 0.$$

Then using Proposition 3.2 we get for any $(\lambda, t) \in (\lambda_m^+ - \eta, \lambda_m^+ + \eta) \times (-\eta, \eta)$, with $\eta > 0$,

$$F_2(\lambda, t) = -\Big(a_m(b)(\lambda - \lambda_m^+) + c_m(b)(\lambda - \lambda_m^+)^2 + d_m(b)t^2 + \big((\lambda - \lambda_m)^2 + t^2\big)\epsilon(\lambda, t)\Big)\mathbb{W}_m$$

with

$$\begin{aligned}
a_m(b) &= -\frac{m[(m\lambda_m^+ - m + 1)^2 - b^{2m}]}{b^{m-1}[(m\lambda_m^+ - m + 1)^2 + b^{2m}]^{\frac{1}{2}}}, \\
c_m(b) &= -\frac{2m^2 b^{1-m}(m\lambda_m^+ - m + 1)^3}{[(m\lambda_m + -m + 1)^2 + b^{2m}]^{\frac{3}{2}}}
\end{aligned}$$

and

$$d_m(b) = \frac{m}{2}(m-1)b^{3-3m}\frac{(b^{2m-2} - (m\lambda_m^+ - m + 1)^2)^2}{([m\lambda_m^+ - m + 1]^2 + b^{2m})^{\frac{1}{2}}} - \frac{\widetilde{\beta}_m}{2}\mathcal{K}_m.$$

Note that for $b = b_m^*$ we have $\Delta_m = 0$ which implies that

$$\lambda_m^+ = \frac{1+b^2}{2}, \quad m\lambda_m^+ - m + 1 = -b^m.$$

Thus we get

$$a_m(b_m^*) = 0, c_m(b_m^*) > 0.$$

Moreover we can check that for $b = b_m^*$,

$$\widetilde{\beta}_m > 0 \quad \text{and} \quad \mathcal{K}_m < 0$$

which implies in turn that $d_m(b_m^*) > 0$. Those properties on the signs remain true for b close to b_m^*, that is, b belongs to some interval $[b_m, b_m^*]$. In addition we deduce from the identity given in Remark 3.3 that for any $b \in (0, b_m^*)$ we have $a_m(b) > 0$. Indeed,

$$\begin{aligned}
(m\lambda_m^+ - m + 1)^2 - b^{2m} &= \big(m\lambda_m^+ - m + 1 - b^m\big)\big(m\lambda_m^+ - m + 1 + b^m\big)\\
&= \big(\sqrt{\Delta_m} - \sqrt{\Delta_m + b^{2m}} - b^m\big)\big(\sqrt{\Delta_m} + b^m - \sqrt{\Delta_m + b^{2m}}\big)\\
&< 0.
\end{aligned}$$

Set $x_0(b) = \frac{a_m(b)}{2c_m(b)}$ and using the change of variables

$$s = \lambda - \lambda_m^+ + x_0(b), \quad \psi(s,t) \triangleq \epsilon\big(s + \lambda_m^+ - x_0(b), t\big)$$

then the equation of F_2 becomes

$$c_m(b)s^2 + d_m(b)t^2 - \frac{a_m^2}{4c_m(b)} + \big((s - x_0(b))^2 + t^2\big)\psi(s,t) = 0 \tag{4.3}$$

and

$$\lim_{(s,t)\to(x_0(b),0)} \psi(s,t) = 0.$$

Note that if we remove ψ from the second term of this equation we get the equation of a small ellipse centered at $(0,0)$ and of semi-axes $\frac{a_m(b)}{2c_m(b)}$ and $\frac{a_m(b)}{2\sqrt{d_m(b)c_m(b)}}$. Thus taking b close enough to b_m^* one can guarantee that this ellipse is contained in the box $(-\epsilon_0, \epsilon_0)^2$ for which the solutions of the equation F_1 still parametrized by φ. By small perturbation we expect to get a curve of solutions to F_2 which is a small perturbation of the ellipse. To prove rigorously this expectation we start with the change of variable,

$$t = x_0(b)\sqrt{\frac{c_m(b)}{d_m(b)}}\,x \quad \text{and} \quad s = x_0(b)y.$$

Consequently, the equation (4.3) becomes

$$\mathcal{G}(b,x,y) \triangleq x^2 + y^2 - 1 + \frac{1}{c_m(b)}\Big((y-1)^2 + \frac{c_m(b)^2}{d_m(b)}x^2\Big)\widehat{\psi}\big(x_0(b), x, y\big) = 0 \tag{4.4}$$

with

$$\widehat{\psi}(\mu, x, y) \triangleq \epsilon\left(\mu\, y + \lambda_m^+ - \mu, \mu\sqrt{\frac{c_m(b)}{d_m(b)}}x\right).$$

Now we shall characterize the geometric structure of the planar set

$$\widehat{\mathcal{E}}_b \triangleq \big\{(x,y) \in \mathbb{R}^2;\ \mathcal{G}(b,x,y) = 0\big\}.$$

Lemma 4.1. *There exists $b_m \in (0, b_m^*)$ such that for any $b \in (b_m, b_m^*)$ the set $\widehat{\mathcal{E}}_b$ contains a C^1-Jordan curve and the point $(0,0)$ is located inside. In addition, this curve is a smooth perturbation of the unit circle $\mathbb{T}$.*

Proof. We shall look for a curve of solutions lying in the set $\widehat{\mathcal{E}}_b$ and that can be parametrized through polar coordinates as follows

$$\theta \in [0, 2\pi] \mapsto (x, y) = R(\theta)e^{\mathrm{i}\theta}.$$

Now we fix b and introduce the function

$$\begin{aligned}\mathcal{F}(\mu, R(\theta)) &\triangleq R^2(\theta) - 1\\ &- \frac{1}{c_m(b)}\left([R(\theta)\sin\theta - 1]^2 + \frac{c_m(b)^2}{d_m(b)}R^2(\theta)\cos^2\theta\right)\widehat{\psi}\Big(\mu, R(\theta)\cos\theta, R(\theta)\sin\theta\Big).\end{aligned}$$

Then according to (4.4) it is enough to solve

$$\mathcal{F}(\mu, R(\theta)) = 0 \quad \text{and} \quad \mu = x_0(b). \tag{4.5}$$

Recall that the function ϵ is defined in the box $(\lambda, t) \in (\lambda_m^+ - \eta, \lambda_m^+ + \eta) \times (-\eta, \eta)$ for some given real number $\eta > 0$. Thus it is not hard to find an implicit value $\mu_0 >$ such that

$$\mathcal{F} : (-\mu_0, \mu_0) \times \mathcal{B} \to C^1(\mathbb{T})$$

is well-defined and is of class C^1, with $\mathcal{B}$ being the open set of $C^1(\mathbb{T})$ defined by

$$\mathcal{B} = \Big\{R \in C^1(\mathbb{T});\ \|1 - R\|_\infty + \|R'\|_\infty < \frac{1}{2}\Big\}.$$

In addition

$$\mathcal{F}(0, 1) = 0 \quad \text{and} \quad \partial_R\mathcal{F}(0, 1)h = 2h,\ \forall h \in C^1(\mathbb{T}).$$

Thus $\partial_R\mathcal{F}(0,1)$ is an isomorphism and by the implicit function theorem we deduce the existence of $\mu_1 > 0$ such that for any $\mu \in (-\mu_1, \mu_1)$ there exists a unique $R_\mu \in \mathcal{B}$ such that (μ, R_μ) is a solution for $\mathcal{F}(\mu, R_\mu) = 0$. It is worthy to point out that μ_1 can be chosen independent of $b \in [b_m, b_m^*]$ because the coefficients $\frac{1}{c_m(b)}$ and $\frac{d_m(b)}{c_m(b)}$ that appear in the nonlinear contribution of $\mathcal{F}$ are bounded. Now since $\lim_{b \to b_m^*} x_0(b) = 0$ then we deduce the existence of $b_m \in (0, b_m^*)$ such that for any $b \in (b_m, b_m^*)$ the set of solutions for (4.5) is described around the trivial solution $R = 1$ (the circle) by the curve $\theta \in [0, 2\pi] \mapsto R_{x_0(b)}(\theta)e^{\mathrm{i}\theta}$. On the other hand from the definition of $\mathcal{B}$ we deduce that this curve of solutions is contained in the annulus centered at the origin and of radii $\frac{1}{2}$ and $\frac{3}{2}$. It should be also non self-intersecting C^1 loop according to the regularity of the polar parametrization. This achieves the proof of the lemma.

□

Now let us see how to use Lemma 4.1 to end the proof of Theorem 1.2. We have already seen in the subsection 2.3 that the vectorial conformal mapping $\Phi \triangleq \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix}$ that describes the bifurcation curve from $(\lambda_m^+, 0)$ is decomposed as follows

$$\Phi(w) = \begin{pmatrix} 1 \\ b \end{pmatrix} w + tv_m + \varphi(\lambda, tv_m) \quad \text{with} \quad (t, \lambda) \in \mathcal{E}_b,$$

where $\mathcal{E}_b$, has been defined in (4.2). As the relationship between the sets $\mathcal{E}_b$ and $\widehat{\mathcal{E}}_b$ is given through a non degenerate affine transformations then the set $\mathcal{E}_b$ is also a C^1-Jordan curve. In addition the point $(\lambda_m^+ - \frac{a_m(b)}{2c_m(b)}, 0)$ is located inside the curve and recall also that the point $(\lambda_m^+, 0)$ belongs to $\mathcal{E}_b$. This fact implies that the curve intersects necessary the real axis on another point $(\lambda_m, 0) \neq (\lambda_m^+, 0)$. By virtue of (2.10) the conformal mappings at those two points coincide which implies in turn that the bifurcation curve bifurcates also from the trivial solution at the point $(\lambda_m, 0)$. Note that looking from the side $(\lambda_m, 0)$ this curve represents V-states with exactly m-fold symmetry and not with more symmetry; to be convinced see the structure of Φ. However from the local bifurcation diagram close to the trivial solution which was studied in [10] we know that the only V-states bifurcating from the trivial solutions with m-fold symmetry bifurcate from the points $(\lambda_m^+, 0)$ and $(\lambda_m^-, 0)$. Consequently we get $\lambda_m = \lambda_m^-$ and this shows that the bifurcation curves of the $m-$fold V-states merge and form a small loop.

The last point to prove concerns the symmetry of the curve $\mathcal{E}_b$ with respect to the λ axis. This reduces to check that for $(\lambda, t) \in \mathcal{E}_b$ then $(\lambda, -t) \in \mathcal{E}_b$. Indeed, if $D = D_1 \setminus D_2$ is an m-fold doubly-connected V-state, its boundaries are parametrized by the conformal mappings

$$\Phi_j(w) = w\left(b_j + a_{j,1}\,\overline{w}^m + \sum_{n\geq 2} a_{j,n}\overline{w}^{mn}\right),\ a_{j,n} \in \mathbb{R}$$

Hence we find that the vectorial conform mapping Φ admits the decomposition

$$\Phi(w) = \begin{pmatrix} 1 \\ b \end{pmatrix} w + \begin{pmatrix} a_{1,1} \\ a_{2,1} \end{pmatrix} \overline{w}^{m-1} + \Psi(w), \quad \Psi \in \mathcal{X}_m$$

Observe that we have a unique t such that

$$\begin{pmatrix} a_{1,1} \\ a_{2,1} \end{pmatrix} \overline{w}^{m-1} = t v_m(w) + \Psi_1(w), \quad \Psi_1 \in \mathcal{X}_m \tag{4.6}$$

Now we consider $\widehat{D} = e^{\frac{i\pi}{m}} D$, the rotation of D with the angle $\frac{\pi}{m}$. Then the new domain is also a V-state with a m-fold symmetry rotating with the same angular velocity. Thus it should be associated to a point $(\lambda, \tilde{t}) \in \mathcal{E}_b$. Now the conformal parametrization of $\widehat{D}$ is given by

$$\widehat{\Phi_j}(w) = e^{-\frac{i\pi}{m}}\Phi_j(e^{\frac{i\pi}{m}}w) = w\left(b_j + \sum_{n\geq 1}(-1)^n \frac{a_{j,n}}{w^{mn}}\right),\ a_{j,n} \in \mathbb{R}.$$

Thus the vectorial conformal parametrization $\widehat{\Phi} \triangleq \begin{pmatrix} \widehat{\Phi}_1 \\ \widehat{\Phi}_2 \end{pmatrix}$ admits the decomposition

$$\widehat{\Phi}(w) = \begin{pmatrix} 1 \\ b \end{pmatrix} w - \begin{pmatrix} a_{1,1} \\ a_{2,1} \end{pmatrix} \overline{w}^{m-1} + \widehat{\Psi}(w), \quad \widehat{\Psi} \in \mathcal{X}_m$$

In view of (4.7) we deduce that

$$\widehat{\Phi}(w) = \begin{pmatrix} 1 \\ b \end{pmatrix} w - t v_m(w) + \widehat{\Psi_1}(w), \quad \widehat{\Psi_1}(w) \triangleq -\Psi_1 + \widehat{\Psi} \in \mathcal{X}_m \tag{4.7}$$

This shows that $(\lambda, -t)$ belongs to the curve $\mathcal{E}_b$ and this concludes the desired result.

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